Parameters of the standard model.

The standard model has 3 basic building blocks:
i) The gauge groups and the associated gauge bosons
ii) The fermions : the basic constituents of matter
iii) The Higgs sector.

i) $SU(3)_C \times SU(2)_L \times U(1)_Y$

8 gluons	W^+, W^-, W^0	B
g_S	or $W^{1,2,3}$; g_2	g_1

$$\mathcal{L}_G = -\frac{1}{4}\left(G_{a\mu\nu} G^{a\mu\nu} + F_{i\mu\nu} F^{i\mu\nu} + B_{\mu\nu} B^{\mu\nu} \right)$$
+ θ terms ;

$$F^i_{\mu\nu} = \partial_\mu W^i_\nu - \partial_\nu W^i_\mu + g_2 \varepsilon^{ijk} W^j_\mu W^k_\nu$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_3 f^{abc} A^b_\mu A^c_\nu$$
alternative: matrix notation

and if acting on a fermion

$$D_\mu \Psi = \left(\partial_\mu - i \frac{g_S}{2} \lambda^a A^a_\mu - i \frac{g_2}{2} \tau^i W^i_\mu - i \frac{g_1}{2} Y_W B_\mu \right) \Psi$$

colour indices — weak isospin indices — hyperdharge

so up to here, 3 parameters.
Gauge bosons transform as : $G = G^a_\mu \frac{\lambda^a}{2} \to U G U^\dagger - \frac{i}{g_3} \partial_\mu U U^\dagger$

ii) Fermions: these are distinguished by their transformations under the 3 gauge groups.

	$SU(3)_c$	$SU(2)_L$	$U(1)$	
q_L =	3	2	1/3	quarks
u_R =	3	1	4/3	
d_R =	3	1	-2/3	
l_L =	1	2	-1	leptons
e_R =	1	1	-2	
(ν_R) =	1	1	0	

the gauge symmetries do not allow for mass terms*

The gauge symmetry are:

$$\Psi \to U\Psi$$

and from the transformation given on the previous page one can check

$$D_\mu \Psi \longrightarrow U D_\mu \Psi$$

The fermions do not introduce any new parameters*

$$q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix} \qquad l_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$$

and each fermion comes in 3 varieties (generations)

u =	up(u)	charm (c)	top (t)
d =	down (d)	strange (s)	bottom (b)
e =	electron (e)	muon (μ)	tau (τ)
ν =	ν_e	ν_μ	ν_τ

$$\mathcal{L}_F = i\,\bar{\Psi}\,\gamma_\mu D^\mu \Psi$$

at this level all couplings are given by the gauge symmetries, so all interactions are identical: quark-lepton universality.

We still have 3 parameters.

*The exception is that a mass term $m_R \overline{\nu_R^c} \nu_R$ is not forbidden by any symmetries

iii) The Higgs sector.

In this sector practically all parameters of the standard model are hidden. I will concentrate on the Higgs in its most simple form. A single doublet with <u>no</u> strong interactions. We believe $SU(3)_c$ to be unbroken

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \quad \text{is} \quad SU(2)_L \text{ doublet and has } Y_W = +1$$

The kinetic term is $\frac{1}{2}\left(D_\mu \Phi\right)^\dagger D^\mu \Phi$

Then there are interaction terms

$$-\mu^2 \Phi^\dagger \Phi + \lambda \left(\Phi^\dagger \Phi\right)^2$$

2 new parameters

Now define $\tilde{\Phi} = i\,\tau_2 \Phi^* = \begin{pmatrix} \phi^{0*} \\ -\phi^- \end{pmatrix}$ is also a doublet with $Y_W = -1$

$\tilde{\Phi}$ is the charge conjugate of Φ

Most parameters are hidden in

$$-\bar{q}^i_L \tilde{\Phi}\, u^j_R M^{ij}_u - \bar{q}^i_L \Phi\, d^j_R M^{ij}_D - \bar{l}^i_L \Phi\, e^j_R M^{ij}$$

$+ h.c.$

(no ν_R assumed here)

at first sight this implies $2 \times 3 \times 9 = 54$ new parameters but not all of these are observable in the standard model.

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Proof of

$$\begin{pmatrix} \lambda_e & & \\ & \lambda_\mu & \\ & & \lambda_\tau \end{pmatrix} = V_{1l} M_L V_{2l}^\dagger$$

$$\Downarrow$$

~~[illegible]~~ $MM^\dagger$, $M^\dagger M$ are both Hermitian => can be diagonalised

$$V_{1l} M M^\dagger V_{1l}^\dagger = \begin{pmatrix} \lambda'_e & & \\ & \lambda'_\mu & \\ & & \lambda'_\tau \end{pmatrix}$$

$$V_{2l} M^\dagger M V_{2l}^\dagger = \begin{pmatrix} \lambda''_e & & \\ & \lambda''_\mu & \\ & & \lambda''_\tau \end{pmatrix}$$

and $\begin{cases} \lambda_e, \lambda_\mu, \lambda_\tau \\ \lambda'_e, \lambda'_\mu, \lambda'_\tau \end{cases}$ are real and solutions of $\det(MM^\dagger - \lambda' \mathbb{1}) = 0$
$= \det(M^\dagger M - \lambda'' \mathbb{1}) = 0$

$$\lambda'_i = \lambda''_i = \lambda_i \lambda_i^* \quad \text{(no sum)}$$

you can always change the phase of λ_i by adding $V_{1l} \to V_{1l} \begin{pmatrix} e^{i\varphi_1} & & \\ & e^{i\varphi_2} & \\ & & e^{i\varphi_3} \end{pmatrix}$

Inverse of the top $\Downarrow$ works if V_{1L}, V_2 at the bottom are unique!

Standard form of mixings and masses

Any matrix can be brought into a diagonal form with real elements on the diagonal by 2 unitary matrices.
So we set:

$$\begin{pmatrix} \lambda_e & & \\ & \lambda_\mu & \\ & & \lambda_\tau \end{pmatrix} = V_{1l}^{*} M_L V_{2l}^{\dagger}$$

and similarly for the up and down masses.
But we now have to follow through what happens in the rest of the Lagrangian:

we now define
$$V_{1l}^{\dagger\, ji} \, l^i_{L_{mass}} = l^j_{L_{gauge}}$$
$$V_{2l}^{\dagger\, ji} \, e^i_{R\,mass} = e^i_{R\,gauge}$$

and the mass term becomes

$$- m_e \bar{e} e - m_\tau \bar{\tau}\tau - m_\mu \bar{\mu}\mu \quad : 3 \text{ new parameters}$$

with $\langle \phi_0 \rangle = \frac{v}{\sqrt{2}}$ $\qquad m_e = \lambda_e \frac{v}{\sqrt{2}}$

and the effect of V_1 and V_2 disappears in the kinetic terms.
For the quarks life is more difficult since we can in the same way not have 2 different transformations on q_L.
By convention we similarly remove M_u into $\lambda_u, \lambda_c, \lambda_t$ or equivalently their 3 masses.
We have then as remaining Yukawa couplings

$$-\bar{q}^i_L \tilde{\Phi} u^i_R \lambda^i_u - \bar{l}^i_L \bar{\Phi} e^i_R \lambda^i_l - \bar{q}^i_L \underbrace{V_{1u}^{*} V_{1d}^{\dagger}}_{U^{*}} \begin{pmatrix} \lambda_d & & \\ & \lambda_s & \\ & & \lambda_b \end{pmatrix}^{ij} \Phi d^j_R$$

so we now have 9 parameters + U

A general 3 x 3 unitary matrix contains 18 − 9 = 3 angles + 6 phases of these phases 5 can be absorbed by redefining the quark fields so U contains 3 angles + one phase.
We now define

$$d^i_{L\,gauge} = U\, d^i_{L\,mass} \qquad . \; U = \text{Kobayashi-Maskawa Matrix}$$

so that the mass term is diagonal but then it reappears in the couplings to W's and remains in the Yukawa couplings of $\bar{u}^i_L$ to d^i_R.
So in gauges other than unitary needs to taken along
So we have 3 gauge couplings
μ, λ or v, m_H or m_Z, m_H
9 fermion masses
3 mixings
1 CP violating parameter.
+ θ + gravitational coupling = 20 parameters

<u>Note:</u>

$$W^{\pm}_\mu = \frac{1}{\sqrt{2}}\left(W^1_\mu \mp W^2_\mu\right)$$

$$Z_\mu = \cos\theta_W\, W^3_\mu - \sin\theta_W\, B_\mu$$

$$A_\mu = \sin\theta_W\, W^3_\mu + \cos\theta_W\, B_\mu$$

$$\tan\theta_W = \frac{g_1}{g_2} \qquad \frac{M_W}{M_Z} = \cos\theta_W$$

$$v = \sqrt{\frac{\mu^2}{\lambda}} \qquad q = I_3 + \frac{1}{2} Y_W$$

$$M_W = \frac{v}{2} g_2$$

$$M_Z = \frac{v}{2}\sqrt{g_1^2 + g_2^2}$$

$$M_\gamma = 0$$

For later use:

$S = :\bar{\psi}\psi:$
$P = i\,\bar{\psi}\gamma_5\psi$
$V_\mu = \bar{\psi}\gamma_\mu\psi$
$A_\mu = \bar{\psi}\gamma_\mu\gamma_5\psi$

C	P	T
S	S	S
P	$-P$	$-P$
$-V_\mu$	V^μ	V_μ
A_μ	$-A^\mu$	A_μ
x^μ	$x^0, -x^i$	$-x^0, x^i$

$P\psi P^{-1} = \gamma^0\psi(x^0,-x^i)$
$T\psi T^{-1} = i\gamma^1\gamma^3\psi(-x^0,x^i)$
$C\psi C^{-1} = i\gamma^2\gamma^0\bar{\psi}^T(x) = i\gamma^2\psi^*(x)$

$$U = \begin{pmatrix} c_1 & -s_1c_3 & -s_1s_3 \\ s_1c_2 & c_1c_2c_3 - s_2s_3e^{i\delta} & c_1c_2s_3 + s_2c_3e^{i\delta} \\ s_1s_2 & c_1s_2c_3 + c_2s_3e^{i\delta} & c_1s_2s_3 - c_2c_3e^{i\delta} \end{pmatrix} \quad \text{(KM)}$$

$$= \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \quad \text{(general)}$$

$$= \begin{pmatrix} 1-\frac{\lambda^2}{2} & \lambda & \lambda^3 A(\rho - i\eta) \\ -\lambda & 1-\frac{\lambda^2}{2} & \lambda^2 A \\ \lambda^3 A(1-\rho-i\eta) & -\lambda^2 A & 1 \end{pmatrix} + O(\lambda^4) \quad \text{(Wolfenstein)}$$

A majorana mass term

We have a second Lorentz invariant combination possible for a mass term.
This is using $\nu_R^c = i\gamma_2 \nu_R^*$

$$\mathcal{L}_{Majorana} = -\frac{m}{2} \overline{\nu_R^c} \nu_R + h.c.$$

with m in general a complex number.
In terms of the Dirac components $\nu_{R\alpha}$ of ν_R this can be written as

$$\mathcal{L}_{Majorana} = -\frac{m}{2} \nu_{R\alpha} S_{\alpha\beta} \nu_{R\beta} + h.c.$$

where $S_{\alpha\beta}$ is a symmetric real 4 by 4 Dirac matrix.

The mass m is in general a complex number but can always be made real by redefining ν_R by a complex phase.

Now in neutrino physics we can make several different ansatzes.

1) Only ν_L^i exist but they can have a most general Majorana mass.
Then the lepton mass term becomes

$$- M_{DL}^{ij} \, \bar{l}_L^i \, \Phi \, e_R^j - \frac{1}{2} M_{ML}^{ij} \, \overline{\nu_L^c}^i \nu_L^j$$

with M^{ij}_{DL} a general complex matrix and M^{ij}_{ML} a complex symmetric matrix.

__Exercise:__ show that in this case we have as free parameters
- 3 charged lepton masses
- 3 neutrino masses
- 3 mixing angles
- 3 phases

The matrix with the angles/phases is the "Pontecorvo-Maki-Nakagawa-Sakata" matrix (other than CKM)

Hint: a general complex symmetric matrix can be diagonalised with real elements on the diagonal by a unitary transformation as

$$M = U^T M_D U$$

(notice the transpose)

2) A certain number of ν_R exist (sterile or right handed neutrinos)
In this case the most general lepton mass matrix is given by

$$- M^{ij}_{DL} \bar{l}^i_L \Phi e^j_R - M^{ij}_{DN} \bar{l}^i_L \tilde{\Phi} \nu^j_R - \frac{1}{2} M^{ij}_M \left(\overline{\nu^c_R}^i \nu^j_R\right)$$

$+ h.c.$

The full analysis in this case is a lot more complicated. I will solve it after explaining the seesaw effect in a one-flavour setting.
Note: case 1 breaks $SU(2)_L \times U(1)_Y$ explicitly but is a very good approximation to the so-called seesaw mechanism case (see below)

The see-saw effect.

In the case of one generation the mass terms in the presence of a right-handed neutrino allowed by $SU(3)_c \times SU(2)_L \times U(1)_Y$ gauge invariance are

$$- g_\ell \; \bar{l}_L \, \Phi \, e^-_R - g_D \; \bar{l}_L \, \tilde{\Phi} \, \nu_R - \frac{m}{2} \, \overline{\nu^c_R} \, \nu_R$$

$+ \; h.c.$

- g_ℓ can be made real by redefining the phase of e^-_R and $m_\ell = \frac{g_\ell v}{\sqrt{2}}$
- g_D can be made real by redefining the phase of $\bar{l}_L$ (and adjusting the phase of e^-_R to keep g_ℓ real)
- m can be made real by adjusting the phase of ν_R (and $\bar{l}_L$, e^-_R to keep the others real)

The neutrino part of the mass term now is (with $m_D = \frac{g_D v}{\sqrt{2}}$)

$$- m_D \, \bar{\nu}_L \, \nu_R - \frac{m}{2} \, \overline{\nu^c_R} \, \nu_R = - \frac{1}{2} \begin{pmatrix} \bar{\nu}_L & \nu_R \end{pmatrix} \begin{pmatrix} 0 & m_D \\ m_D & m \end{pmatrix} \begin{pmatrix} \bar{\nu}_L \\ \nu_R \end{pmatrix}$$

where on the r.h.s. I have suppressed all Dirac factors in between.

The latter matrix can be diagonalized by

$$\begin{pmatrix} 0 & m_D \\ m_D & m \end{pmatrix} = U^T \begin{pmatrix} + \frac{m}{2} - \frac{1}{2}\sqrt{m^2 + 4 m_D^2} & 0 \\ 0 & \frac{m + \sqrt{m^2 + 4 m_D^2}}{2} \end{pmatrix} U$$

$$U = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

$$a = \frac{1}{\sqrt{\frac{1}{2}(m^2 + 4 m_D^2) - \frac{m}{2}\sqrt{m^2 + 4 m_D^2}}} \, m_D$$

$$b = \frac{1}{\sqrt{}} \left(\frac{m}{2} - \frac{1}{2}\sqrt{m^2 + 4 m_D^2} \right)$$

In the limit $m >> m_D$ we get

$$\lambda_1 = - \frac{m_D^2}{m} + O\left(\frac{1}{m^3}\right)$$

$$\lambda_2 = m + \frac{m_D^2}{m} + O\left(\frac{1}{m^3}\right)$$

and the mass eigenstates are

$$\nu_1 = \bar{\nu}_L - \frac{m_D}{m} \nu_R + O\left(\frac{1}{m^2}\right)$$

$$\nu_2 = \frac{m_D}{m} \bar{\nu}_L + \nu_R + O\left(\frac{1}{m^2}\right)$$

So for the case $m >> m_D$ a very good approximation is to treat it is $\nu_L, \bar{\nu}_L$ being the only degree of freedom with a Majorana mass.
This is the general case described on page ① for 3 generations.

The general case

Start with

$$-\bar{l}^{\,i}_L M^{ij}_{DL} \Phi_I e^j_R - \bar{l}^{\,i}_L M^{ij}_{DN} \tilde{\Phi}_I \nu^j_R - \frac{1}{2} M^{ij}_M \bar{\nu^c}^i \nu^j$$

We can diagonalize M^{ij}_{DL} via $U^{\dagger}_{DL} M_{DLD} V_{DL}$

and the neutrino mass matrix is now

$$-\frac{1}{2} \left(\bar{\nu}^1_L \; \bar{\nu}^2_L \; \bar{\nu}^3_L \; \nu^1_R \; \nu^2_R \; \nu^3_R \right) \begin{pmatrix} 0 & M_D \\ M_D^T & M_M \end{pmatrix} \begin{pmatrix} \bar{\nu}^1_L \\ \bar{\nu}^2_L \\ \bar{\nu}^3_L \\ \nu^1_R \\ \nu^2_R \\ \nu^3_R \end{pmatrix}$$

and the full 6 by 6 mass matrix can be diagonalized by a 6 × 6 unitary matrix as

$$U^T_M M_{MD} U_M = \begin{pmatrix} 0 & M_D \\ M_D^T & M_M \end{pmatrix}$$

$$U_M = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}$$

with u_1, u_2, u_3, u_4 3 × 3 matrices satisfying

$$\begin{cases} u_1 u_1^{\dagger} + u_2 u_2^{\dagger} = \mathbb{1} \\ u_3 u_3^{\dagger} + u_4 u_4^{\dagger} = \mathbb{1} \\ u_1 u_3^{\dagger} + u_2 u_4^{\dagger} = 0 \end{cases}$$

~~the mass eigenstates are now~~

The weak interaction states interacting with e^-, μ^-, τ^- now are

$$\begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix} = U_{DL} \left(u_1^\dagger \begin{pmatrix} \tilde{\nu}_1 \\ \tilde{\nu}_2 \\ \tilde{\nu}_3 \end{pmatrix} + u_2^\dagger \begin{pmatrix} \bar{\tilde{\nu}}_R^1 \\ \bar{\tilde{\nu}}_R^2 \\ \bar{\tilde{\nu}}_R^3 \end{pmatrix} \right)$$

The mass eigenstates are
$$\begin{pmatrix} \bar{\tilde{\nu}}_1 \\ \bar{\tilde{\nu}}_2 \\ \bar{\tilde{\nu}}_3 \\ \tilde{\nu}_R^1 \\ \tilde{\nu}_R^2 \\ \tilde{\nu}_R^3 \end{pmatrix} = U_M^\dagger \begin{pmatrix} \bar{\nu}_L^1 \\ \bar{\nu}_L^2 \\ \bar{\nu}_L^3 \\ \nu_R^1 \\ \nu_R^2 \\ \nu_R^3 \end{pmatrix}$$

So we have in principle 6 neutrino masses but the mixing matrix is more complicated than just unitary.

We define
$$\begin{cases} \tilde{u}_1 = U_{DL} u_1^\dagger \\ \tilde{u}_2 = U_{DL} u_2^\dagger \end{cases}$$
and these satisfy $\tilde{u}_1^\dagger \tilde{u}_1 + \tilde{u}_2^\dagger \tilde{u}_2 = \mathbb{1}$

$\tilde{u}_1, \tilde{u}_2$ have 36 free parameters and satisfy 9 real relations;
3 phases can be removed by redefining the phases of the charged leptons

We started with 48 parameters in the matrices $M_{DL}^{ij}, M_{DN}^{ij}, M_M^{ij}$

and now have 9 masses plus 24 parameters in the mixing sector

of these 12 are mixing angles

Solution of $\tilde{u}_1^\dagger \tilde{u}_1 + \tilde{u}_2^\dagger \tilde{u}_2 = \mathbb{1}$

$c_d s_1 c_2 \quad c s_{d1} c_2 \quad c s_{d2} \qquad s_d \tilde{c}_1 \tilde{c}_2 \quad s_d \tilde{s}_1 \tilde{c}_2 \quad s_d \tilde{s}_2$

now the $(\mathbb{1})_{11}$ element is secure
the first column needs to have square less than 1

6 x 6 unitary matrix has $\frac{6 \times 5}{2} = 15$ angles

$c_1 \quad s_1 c_2 \quad s_1 s_2 c_3 \quad s_1 s_2 s_3 c_4 \quad s_1 s_2 s_3 s_4 c_5 \quad s_1 s_2 s_3 s_4 s_5$

For <u>real</u> matrices: $\tilde{u}_1^\dagger$ and $\tilde{u}_2$ we have 18 parameters and 6 constraints
$\Rightarrow$ 12 parameters can show up.