Step I : the Lorentz - Poincaré group

a typical rotation
$$\begin{cases} x = \cos\theta\, x - \sin\theta\, y \\ y = \sin\theta x + \cos\theta\, y \end{cases}$$

a typical Lorentztransformation
$$\begin{cases} x = \gamma\,(x - v\,t) \\ t = \gamma\,(t - v\,x) \end{cases}$$

$$\gamma = \frac{1}{\sqrt{1-v^2}} \qquad \text{set} \quad \begin{aligned} \gamma &= \cosh\eta \\ \gamma v &= \sinh\eta \end{aligned} \qquad \gamma^2 - v^2\gamma^2 = 1 \ \text{bingo.}$$

A small difference: $\theta \to 0 \to 2\pi$: compact

$\eta \to -\infty \to +\infty$: non - compact

$J^{\mu\nu} = i\left(x^\mu\partial^\nu - x^\nu\partial^\mu\right)$ are the space time generators of the Lorentzgroup

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\left[g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}\right]$$

6 generators
J^{ij} generate rotations
J^{0i} generate boosts

boosts & rotations do not commute

If you made it compact (Wick rotation) the Lorentzgroup would have been
$SO(4) = SU(2) \times SU(2)$

The Lorentz group also falls apart in 2 parts

The 2 sets are

$$\left(J^{ij} \pm \frac{1}{2}\varepsilon^{ijk} J^{0k}\right) \quad \text{or equivalently} \quad \left(J^{0i} \pm \frac{1}{2}\varepsilon^{ijk} J^{jk}\right)$$

they both form an $SU(2)$ group. ($SU(2)$ like: non compact !)

for fermions $J^{\mu\nu} \to \frac{i}{2}[\gamma^\mu, \gamma^\nu]$

the 2 sets are now $S^{0i} = \frac{-i}{2}\begin{pmatrix} \sigma^i & \\ & \sigma^k \end{pmatrix}$ ← (not Hermitian!)

$$S^{ij} = \frac{1}{2}\varepsilon^{ijk}\begin{pmatrix} \sigma^k & \\ & \sigma^k \end{pmatrix}$$

so under Lorentz transformations

$$\begin{cases} \psi_L \to \left(1 - i\frac{\vec{\theta}\cdot\vec{\sigma}}{2} - \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)\psi_R \\ \psi_R \to \left(1 - i\frac{\vec{\theta}\cdot\vec{\sigma}}{2} + \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)\psi_R \end{cases}$$

so they move in a representation of the separate groups

⇒ the underlying reasons for the dotted and undotted indices

⇒ treat them separately from now on

Step II : 2 component notation

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}$$

$\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}$ (Martin's conventions)

$$\sigma^\mu = (\mathbb{1}, \vec\sigma)$$

$$\bar\sigma^\mu = (\mathbb{1}, -\vec\sigma)$$

$$\psi_D = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} \xi_\alpha \\ \chi^{\dagger\dot\alpha} \end{pmatrix}$$

ξ, χ are both left handed Weyl spinors

$$\bar\psi_D = \left(\chi^\alpha \quad (\xi^\dagger)_{\dot\alpha} \right)$$

$\xi^\dagger\ \chi^\dagger$ are both right handed Weyl spinors

$$(\sigma^\mu)_{\alpha\dot\alpha} \qquad (\bar\sigma^\mu)^{\dot\alpha\alpha}$$

$\varepsilon^{12} = -\varepsilon^{21} = \varepsilon_{21} = -\varepsilon_{12} = 1$ all others zero

$\varepsilon^{\alpha\beta}, \varepsilon_{\alpha\beta}, \varepsilon^{\dot\alpha\dot\beta}, \varepsilon_{\dot\alpha\dot\beta}$; raise & lower indices: (beware of order) $\varepsilon^{\alpha\beta}\xi_\beta$, $\varepsilon_{\alpha\beta}\xi^\beta$

${}^\alpha{}_\alpha$ or ${}_{\dot\alpha}{}^{\dot\alpha}$: Einstein convention or matrix : suppress

$$\bar\psi_D \gamma^\mu \psi_D = \left(\chi^\alpha \quad (\xi^\dagger)_{\dot\alpha} \right) \begin{pmatrix} 0 & \sigma^\mu_{\alpha\dot\beta} \\ \bar\sigma^{\mu\dot\alpha\beta} & 0 \end{pmatrix} \begin{pmatrix} \xi_\beta \\ (\chi^\dagger)^{\dot\beta} \end{pmatrix}$$

$$= \chi^\alpha \sigma^\mu_{\alpha\dot\beta} \chi^{\dagger\dot\beta} + \xi^\dagger_{\dot\alpha} \bar\sigma^{\mu\dot\alpha\beta} \xi_\beta$$

indices must always match !

Intermezzo : Grassman numbers

- $\theta^2 = 0 \qquad \theta_1\theta_2 = -\theta_2\theta_1$ anticommute!
- a different θ is in each spacetime point!
  ξ has 2 different Grassmann numbers in it.
- $\frac{\partial}{\partial\theta}\theta = 1 \qquad \int d\theta\, 1 = 0 \qquad \int d\theta\,\theta = 1$

$$\xi\chi \equiv \xi^\alpha \chi_\alpha = -\chi_\alpha \xi^\alpha = -\chi^\beta \underbrace{\varepsilon_{\alpha\beta}\,\varepsilon^{\alpha\gamma}}_{-\delta_\beta^{\ \gamma}} \xi_\gamma = \chi^\gamma \xi_\gamma = \chi\xi$$

$$\left(\xi^\dagger \bar\sigma^\mu \chi\right) = -\left(\xi\sigma^\mu\chi^\dagger\right)$$

$$\left(\xi\chi\right)^\dagger = \left(\xi\chi\right)^*$$

You can work out a whole list of identities to go with this.

$$\mathcal{L}_D = i\left(\xi^\dagger\bar\sigma^\mu\partial_\mu\xi + \chi^\dagger\bar\sigma^\mu\partial_\mu\chi\right) - M_D\left(\xi\chi + \xi^\dagger\chi^\dagger\right)$$

$$\mathcal{L}_M = i\left(\xi^\dagger\bar\sigma^\mu\partial_\mu\xi\right) - M_M\left(\xi\xi + \xi^\dagger\xi^\dagger\right)$$

$$\chi = \xi\ ! \quad \text{or} \quad \psi_M = \begin{pmatrix} \xi_\alpha \\ (\xi^\dagger)^{\dot\alpha} \end{pmatrix}$$

You can fully rewrite the SM in this way.