Added Notes to chapter 3

Unitarity and Analyticity

S matrix: scattering is described by a series of incoming plane wave states $|f, in\rangle$ and outgoing (final) planes wave states $|f, out\rangle$. $|i, in\rangle$ is a plane wave at $t=-\infty$ but not necessarily at $t=+\infty$ and the other way round for $|f, out\rangle$

The transition probabilities are

$$W_{f\leftarrow i} = |\langle f, out | i, in\rangle|^2$$

The S matrix is defined by : $\langle f, out | i, in\rangle = \langle f, in | S | i, in\rangle$

S is unitary : we don't loose pieces of the universe

$$S = 1 + iT \qquad \text{(definition of } T\text{)}$$

and

$$d\sigma_{2\to n} = \frac{1}{4\sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2}} \int d\tilde{p}_3 \cdots d\tilde{p}_{n+2} \, |\langle p_3 \cdots p_{n+2} | T | p_1 p_2\rangle|^2 \times (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - \cdots - p_{n+2})$$

with the single plane wave states normalized to

$$\langle p_1 | p_2\rangle = 2p^0 (2\pi)^3 \delta^3(\vec{p}_1 - \vec{p}_2)$$

$$d\tilde{p}_n = \frac{d^4 p_n}{(2\pi)^4} \delta(p_n^2 - m_n^2) = \frac{d^3 p_n}{(2\pi)^4\, 2E_n}$$

(next all states are in !)

Now: $S^\dagger S = 1 + i(T - T^\dagger) + T^\dagger T = 1$

$\langle f | S^\dagger S | i \rangle$ gives

$$i\left(T_{fi} - T^*_{if}\right) = -\sum_n (2\pi)^4 \, \delta(P_n - (p_1 + p_2)) \cdot T^*_{fi} \, T_{ni}$$

or the absorptive part of the forward crossection ($f = i$) is related to the total cross section!

and the absorptive part has a known sign!

$$\operatorname{Im} T_{ii} = \lambda^{1/2}(s, m_a, m_b) \, \sigma_{tot}(i)$$

$$\lambda(x, y, z) = (x + y + z)^2 - 4xy - 4yz - 4xz$$

and we defined $\quad T_{fi} \, (2\pi)^4 \, \delta^4(P_f - P_i) = \langle f | T | i \rangle$

To fully exploit the consequences we need to analytically continue the amplitudes (the T_{if}) to complex values of the momenta.

Let me present the argument in simple perturbation theory; all quantities are given by a series like:

$$\psi(E) = \psi_0(E) + \sum_n \frac{1}{E - E_{0n}} H_i | \psi_n(E) \rangle + \dots$$

now adding the $i\varepsilon$ prestription and allowing E to become complex then shows that all amplitudes are analytic in the full E-plane

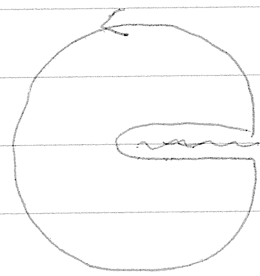
and the $i\varepsilon$ prescription gives the correct absorptive parts whenever there are real intermediate states possible.

Subtractions etc.

We have a typical function $\Pi(s)$ and we now want to make the dispersion relation convergent. I.e. the contribution of the circle at ∞ should be made to vanish.
This can be done in various ways.
We can use $\frac{\Pi(s)}{s(s\,\cancel{}-q^2)}$ instead of $\frac{\Pi(s)}{s(s-q^2)}$

Then we have

$$\frac{1}{2\pi i}\oint ds\,\frac{\Pi(s)}{s-q^2} = \Pi(q^2) - \frac{1}{\pi}\int_0^\infty ds\,\frac{\text{Im}\,\Pi(s)}{s-q^2}$$



instead of zero since the function now has a pole at $s=q^2$ with residue q^2. This is in fact what people usually call a dispersion relation
If the circle at ∞ still does not vanish we can subtract the dispersion relation.
Then we use $\frac{q^2\,\Pi(s)}{s(s-q^2)}$ instead and after using

$$\frac{q^2}{s(s-q^2)} = \frac{1}{s-q^2} - \frac{1}{s}$$

we obtain instead

$$\Pi(q^2) = \Pi(0) + \frac{1}{\pi}\int_0^\infty ds\,\frac{q^2}{s(s-s_0)}\,\text{Im}\,\Pi(s)$$

Exercise: what happens if we need to subtract twice at zero?

i.e. use $\frac{q^4\,\Pi(s)}{s^2(s-q^2)}$

3. Precise predictions from QCD at low energies: τ decays and Weinberg sum rules

QCD is a field theory => all calculable quantities satisfy the general theorems of quantum field theory: unitarity and analyticity
A simple statement of these quantities is:
The singularity structure of amplitudes and Green functions consists of poles and cuts and only for values of the kinematical variables such that intermediate on-shell states are possible.
The use of this allows to use the general theory of functions of complex variables to provide constraints on the functions together with its behaviour from perturbative QCD at large masses/momenta.
A powerful addition was the use of the operator product expansion to add effects from the "nonperturbative" QCD vacuum (SVZ)
As a first example let me derive the Weinberg sum rules.
These are about the quantities

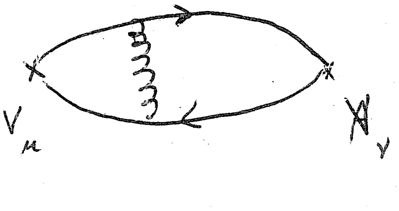
$$\Pi^{V,A}_{\mu\nu}(q^2) = \Pi^{(0)}_{V,A}(q^2)\, q_\mu q_\nu + \Pi^{(1)}_{V,A}(q^2)\,(q_\mu q_\nu - q^2 g_{\mu\nu})$$

$$= i\int d^4x\, e^{iq\cdot x} \langle 0|\, T\begin{pmatrix} V_\mu(x)\, V_\nu(0) \\ A_\mu(x)\, A_\nu(0)\end{pmatrix} |0\rangle$$

In the limit where the current quark masses are zero we have

$$\Pi^{(0)}_{V,A} = 0$$

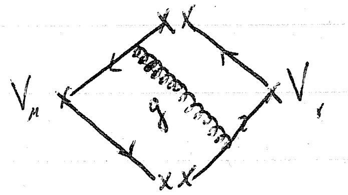
At high energies we can calculate perturbatively



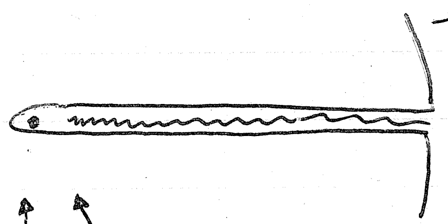
all diagrams are identical

$$\Rightarrow \quad \Pi^{(1)}_V(q^2) - \Pi^{(1)}_A(q^2) = 0$$

The leading contribution is (to the difference!) (*)



$$\propto \alpha_s \frac{\langle \bar{q}q\bar{q}q \rangle}{Q^6}$$



Now $\displaystyle \frac{1}{2\pi i} \int_C \left[\Pi_V^{(1)}(q^2) - \Pi_A^{(1)}(q^2) \right] (q^2)^\alpha \, dq^2$

$$= + \frac{1}{2\pi i} \oint \frac{2 f_\pi^2 (q^2)}{q^2} q^{2\alpha} dq^2 + \int_0^\infty dq^2 \, \frac{1}{\pi} \operatorname{Im} \left(\Pi_V^{(1)} - \Pi_A^{(1)} \right) \cdot$$

$$+ \frac{1}{2\pi i} \int_{C'} q^{2\alpha} \left(\Pi_V^{(1)} - \Pi_A^{(1)} \right) dq^2$$

From (*) the circle integral vanishes for $\alpha \leq 1$

$\alpha = 0$ gives $\displaystyle 2 f_\pi^2 = \int_0^\infty dq^2 \, \frac{1}{\pi} \operatorname{Im} \left(\Pi_V^{(1)} - \Pi_A^{(1)} \right)$

$\alpha = 1$ gives $\displaystyle \int_0^\infty dq^2 \, \frac{q^2}{\pi} \operatorname{Im} \left(\Pi_V^{(1)} - \Pi_A^{(1)} \right) = 0$

Saturation by vector + axial vector gives:

$$f_V^2 m_V^2 - f_A^2 m_A^2 = f_\pi^2 \quad \text{and} \quad f_V^2 m_V^4 = f_A^2 m_A^4$$

$\mathrm{Im}\,\Pi_V^{(1)}$ can be measured in $e^+e^- \to$ hadrons
$\mathrm{Im}\,\Pi_A^{(1)}$ " " $\tau \to \nu_\tau +$ hadrons.

The results are in fact very well described by the Weinberg sum rules

A second application of this type of sum rules is the calculation of

$$R_\tau = \frac{\Gamma(\tau^- \to \nu_\tau \text{ hadrons}(\gamma))}{\Gamma(\tau^- \to \nu_\tau e^- \bar{\nu}_e(\gamma))}$$

experimentally this can be described via

$$R_\tau^B = \frac{1 - B_e - B_\mu}{B_e}$$ (B_e is the branching ratio into electrons)

$$\alpha = \frac{\Gamma_\tau - \Gamma_{\tau\to e} - \Gamma_{\tau\to\mu}}{\Gamma_{\tau\to e}}$$

$\Gamma_{\tau\to e,\mu}$ is purely leptonic and can thus be very well calculated.

R_τ theoretically:

using the 2-point functions defined above

$$R_\tau = 12\pi \int_0^{M_\tau^2} \frac{ds}{M_\tau^2}\left(1 - \frac{s}{M_\tau^2}\right)^2\left[\left(1 + \frac{2s}{M_\tau^2}\right)\mathrm{Im}\,\Pi^{(1)} + \mathrm{Im}\,\Pi^{(0)}\right]$$

$$\Pi^{(i)} = |V_{ud}|^2\left(\Pi_{ud,V}^{(i)} + \Pi_{ud,A}^{(i)}\right) + |V_{us}|^2\left(\Pi_{us,V}^{(i)} + \Pi_{us,A}^{(i)}\right)$$

Notice that we can also calculate various moments of the decay rate this way.

This can now be rewritten in a circle integral over the circle at $|s| = M_\tau^2$

$$R_\tau = 6\pi i \oint_{|s|=M_\tau^2} \frac{ds}{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^2 \left[\left(1 + 2\frac{s}{M_\tau^2}\right) \Pi^{(1)} + \Pi^{(0)}\right]$$

which does not involve any small s and can be fully evaluated in perturbative QCD.
It becomes

$$R_\tau^{pert} = 3\left(|V_{ud}|^2 + |V_{us}|^2\right)\left(1 + a + 5.2023\, a^2 + 26.366\, a^3 + \dots + \delta^{nonpert.}\right)$$

$$a = \frac{\alpha_s(M_\tau)}{\pi}$$

The nonperturbative contributions are from the condensates:

$$\delta^{nonpert} = -8\left(1 + \frac{16}{3}\frac{\alpha_s}{\pi}\right)\frac{m_i^2 + m_j^2}{M_\tau^2} + \frac{11}{4}\pi^2 a^2 \frac{\langle \alpha_s/\pi\, G^2\rangle}{M_\tau^4} + \dots$$

these are here the leading theoretical uncertainty

eg for $\alpha_s(M_\tau) = 0.34$ one gets $R_\tau = 3.62 \pm 0.06$

This leads to an extremely nice test of QCD running of α_s when brought down from the LEP to M_τ

An example: f_V and M_V from QCD sum rules.

Let us look at $j_\mu = \bar{u}\gamma_\mu d$

$$i\int d^4x\, e^{iq\cdot x} \langle T(j_\mu(x)\, j_\nu^\dagger(0))\rangle$$

$$= (q_\mu q_\nu - q^2 g_{\mu\nu})\left\{-\frac{1}{4\pi^2}(1+\alpha_s/\pi)\ln\frac{Q^2}{\mu^2} + \dots\right\}$$

(p 396 Shifman book)

$$\Pi_V^{(1)}(q^2) = -\frac{1}{4\pi^2}\left(1+\frac{\alpha_s}{\pi}\right)\ln\frac{-q^2-i\varepsilon}{\mu^2} \quad \text{for large } Q^2$$

$$\operatorname{Im}\Pi_V^{(1)}(q^2) = \frac{1}{4\pi^2}\left(1+\frac{\alpha_s}{\pi}\right)\pi$$

$$\Pi(q^2) = \frac{1}{\pi}\int_0^\infty \frac{ds}{s-q^2}\,\frac{1}{4\pi}\left(1+\frac{\alpha_s}{\pi}\right) \quad \text{(up to constants; so it fits)} \qquad (1)$$

Now we choose an <u>ansatz</u> for $\operatorname{Im}\Pi$

$$\operatorname{Im}\Pi_V^{(1)}(s) = 2\pi f_V^2 M_V^2\,\delta(s-M_V^2) + \theta(s-s_0)\,\frac{1}{4\pi}(1+\alpha_s/\pi)$$

now one can use the procedure of Borel transformation:

apply
$$\hat{L}_M = \lim_{\substack{-q^2 \to +\infty \\ n \to \infty \\ -q^2/n = M^2 \text{ fixed}}} \frac{1}{(n-1)!} (-q^2)^n \left(\frac{d}{dq^2}\right)^n$$

to both sides of (1). Notice that this procedure removes all subtraction constants.

Notice that
$$\left(\frac{d}{dq^2}\right)^n \frac{1}{s-q^2} = n! \frac{1}{(s-q^2)^{n+1}}$$

so
$$\hat{L}_M \left(\frac{1}{s-q^2}\right) = \lim_{n \to \infty} \frac{n!}{(n-1)!} \left(\frac{nM^2 \cancel{n}}{s + nM^2 \cancel{n}}\right)^{n+1} \frac{1}{nM^2}$$

$$= \lim_{n \to \infty} \cancel{n} \frac{1}{\left(1 + \frac{s}{nM^2}\right)^{n+1}}$$

$$\simeq \frac{1}{M^2} e^{-s/M^2}$$

$$\hat{L}_M \left(\ln(-q^2)\right) = \lim_{n \to \infty} \frac{(nM^2)^n}{(n-1)!} (-1)^{n-1} \frac{(n-1)!}{(nM^2)^n} = -1$$

So we get

$$\frac{1}{4\pi^2}(1 + \alpha_s/\pi) = \frac{2 f_V^2 M_V^2 e^{-M_V^2/M^2}}{M^2} + \frac{1}{4\pi^2}\left(1 + \frac{\alpha_s}{\pi}\right) e^{-s_0/M^2}$$

Choose $M^2 = M_V^2$ and we get

$$2 f_V^2 e^{-1} = \frac{1}{4\pi^2}\left(1 + \frac{\alpha_s}{\pi}\right)$$

or $$f_V^2 \approx \frac{e}{8\pi^2} \approx 0.034$$

$$f_V \approx 0.18$$

experimental value is 0.20

We can now look at the same sum rule for $s\Pi(s)$

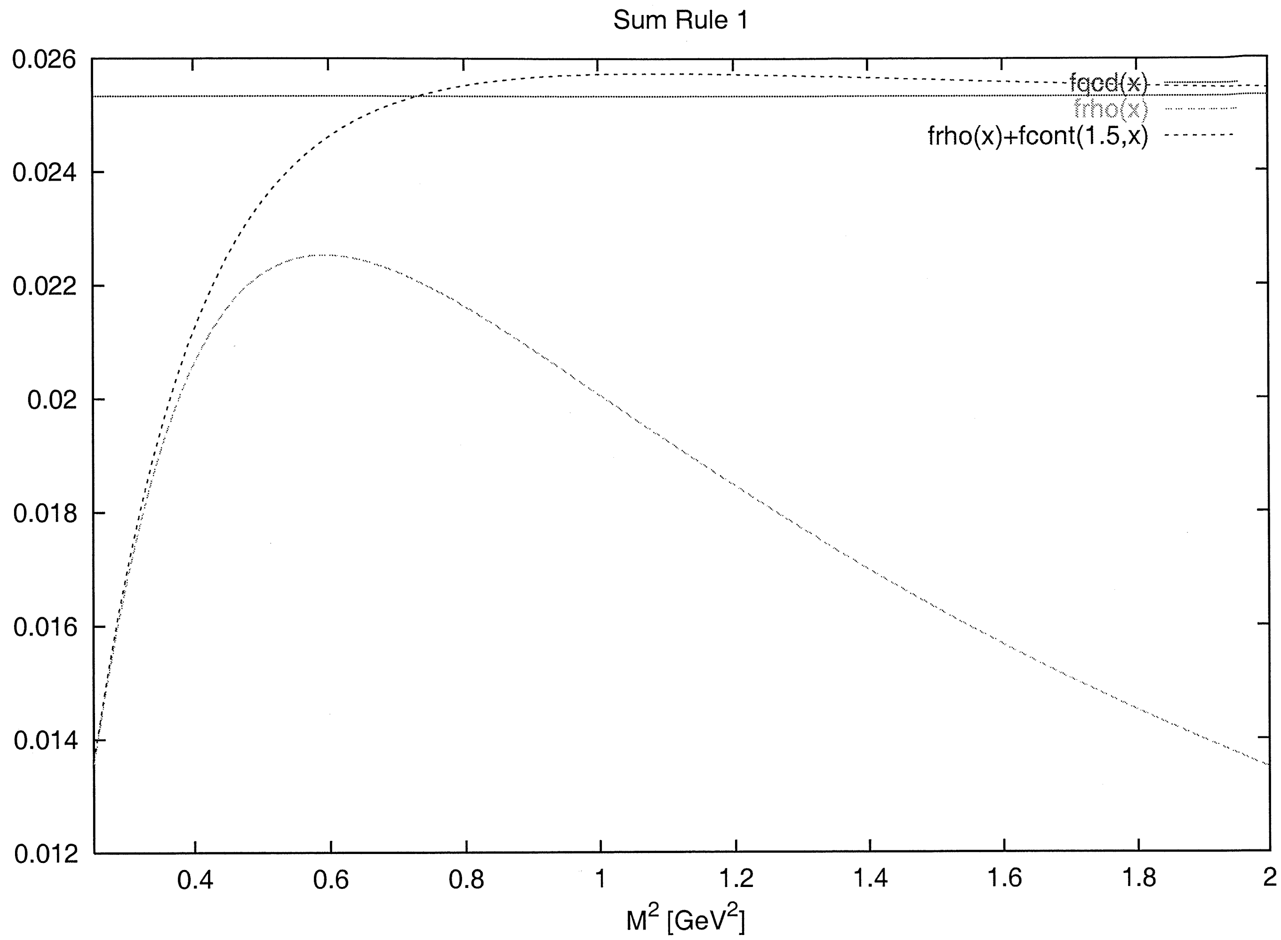
using $\hat{L}_M\left(q^2 \ln(M^2 - q^2)\right) = -M^2$

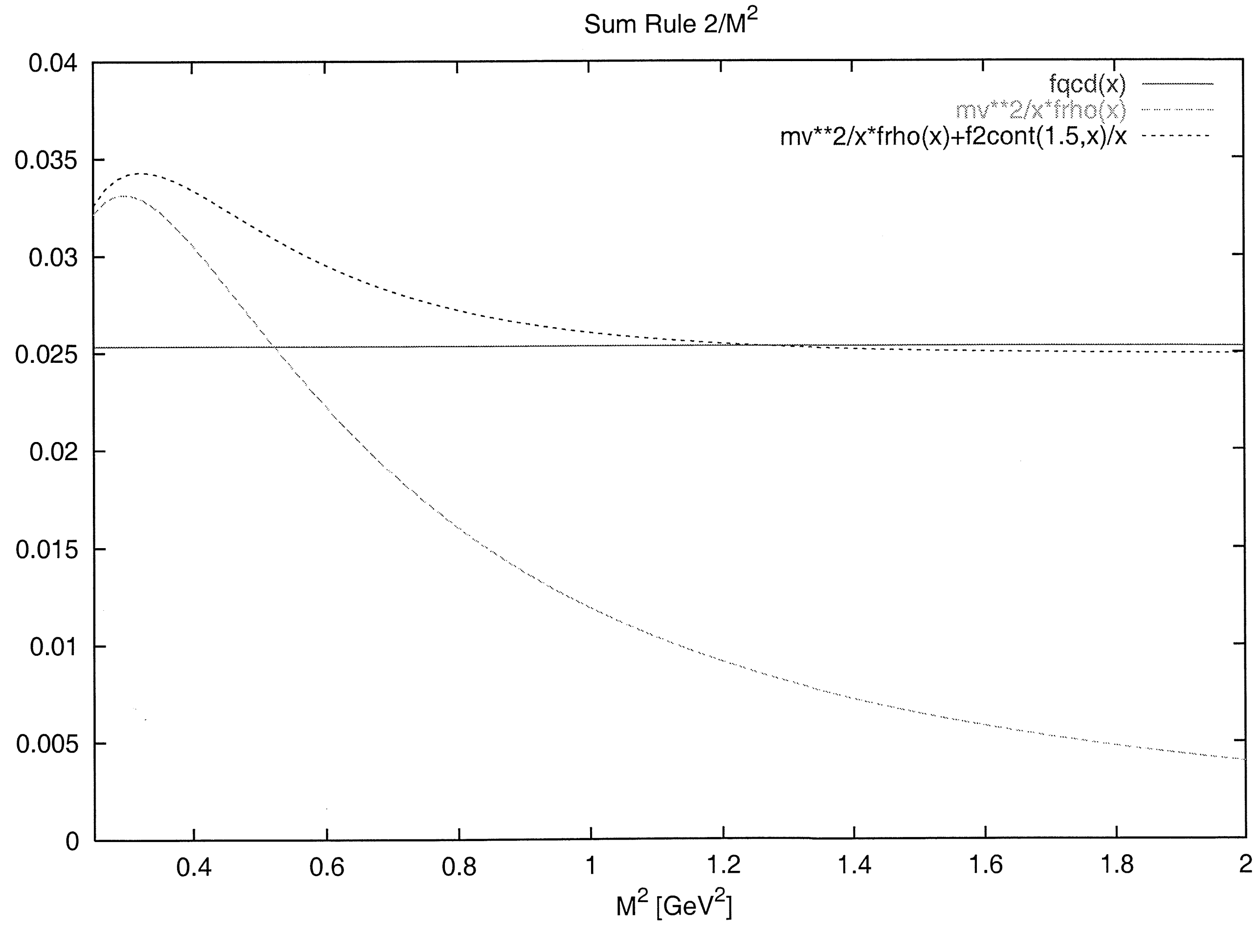
we obtain

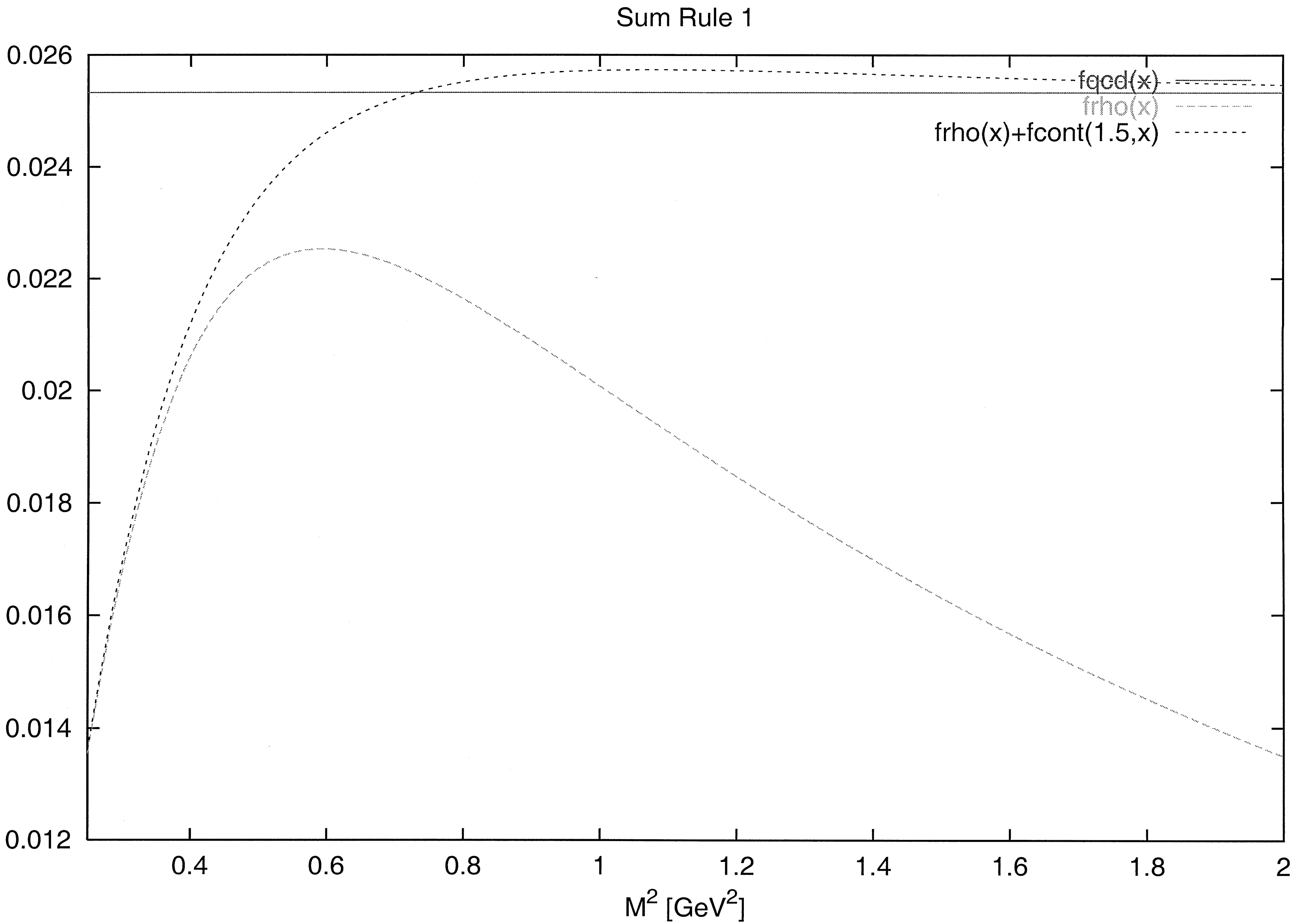
$$\frac{M^2}{4\pi^2}\left(1 + \frac{\alpha_s}{\pi}\right) = \frac{2 f_V^2 M_V^4 e^{-M_V^2/M^2}}{M^2} + \frac{M^2}{4\pi^2}\left(1 + \frac{\alpha_s}{\pi}\right)\left(1 + \frac{s_0}{M^2}\right)e^{-s_0/M^2}$$

we can now check how well these two sum rules work
(choosing $s_0 \approx 1.5\ GeV^2$)
as a function of M^2

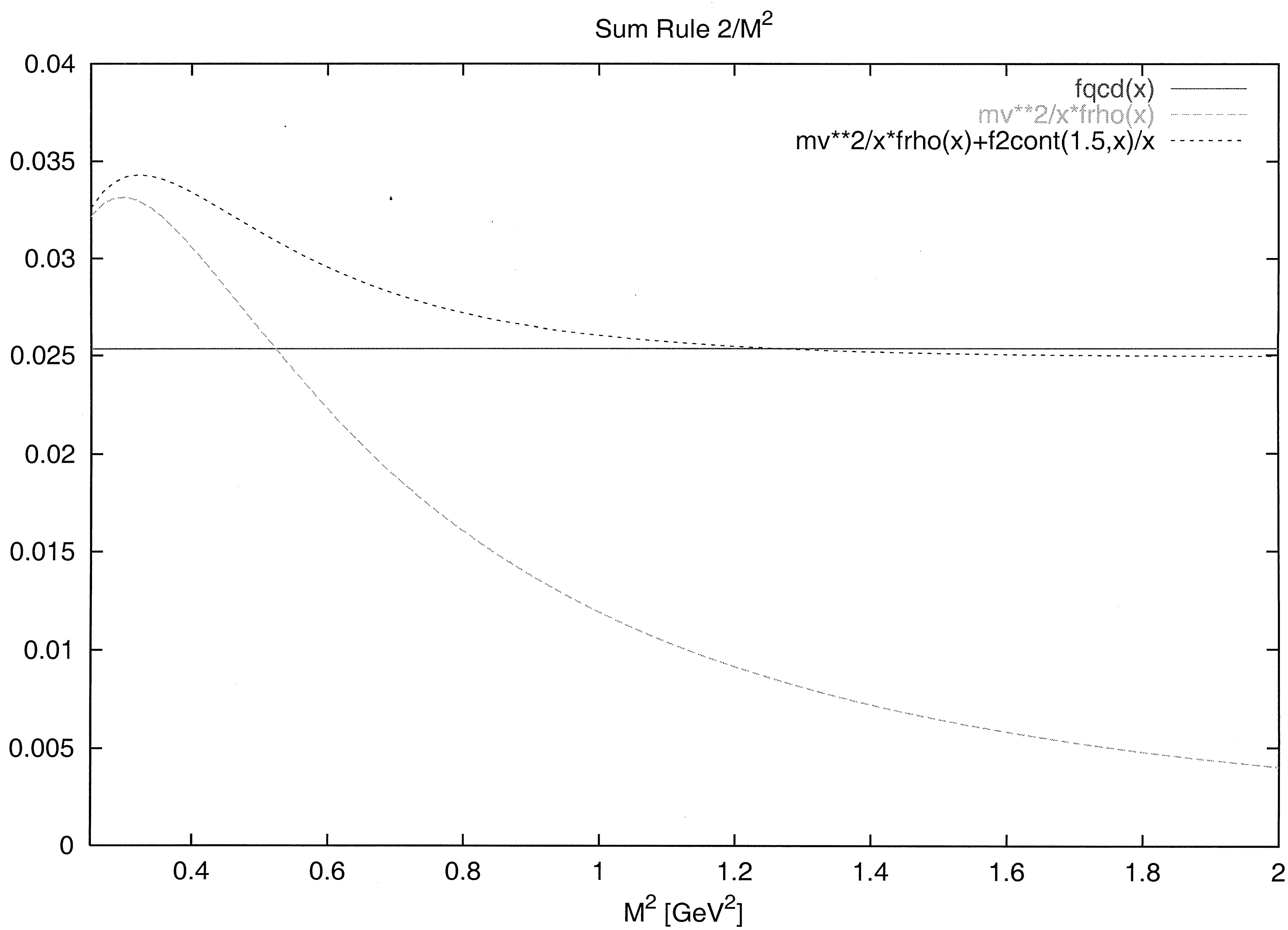
Notice we get so called stability plateaus in both sum rules.


Sum Rule 1
0.026
0.024
0.022
0.02
0.018
0.016
0.014
0.012
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
M^2 [GeV^2]
fqcd(x)
frho(x)
frho(x)+fcont(1.5,x)


Sum Rule 2/M^2
0.04
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
M^2 [GeV^2]
fqcd(x)
mv**2/x*frho(x)
mv**2/x*frho(x)+f2cont(1.5,x)/x



Sum Rule 1
0.026
0.024
0.022
0.02
0.018
0.016
0.014
0.012
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
M^2 [GeV^2]
fqcd(x)
frho(x)
frho(x)+fcont(1.5,x)



Sum Rule 2/M2
fqcd(x)
mv**2/x*frho(x)
mv**2/x*frho(x)+f2cont(1.5,x)/x
0.04
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
M2 [GeV2]

A full calculation of: $\Pi^{V}_{\mu\nu}$

$$\Pi^{V}_{\mu\nu} \equiv i \int d^4x \; e^{iq\cdot x} \langle 0 | T(J_\mu(x) J^\dagger_\nu(0)) | 0 \rangle$$

with $J_\mu(x) = \bar{u} \gamma_\mu d(x)$

and we will for the moment neglect the colour degree of freedom and give u and d the same mass.

$$\Pi^{V}_{\mu\nu} = i \int d^4x \; e^{iq\cdot x} \langle 0 | T\big(\bar{u}_\alpha(x) \gamma_{\mu\alpha\beta} d_\beta(x) \bar{d}_\gamma(0) \gamma_{\nu\gamma\delta} u_\delta(0) \big) | 0 \rangle$$

This can be evaluated using Wicks theorem to rewrite a time-ordered product (T) into normal ordered products and remembering that it is a matrix element between vacuum states.
Only two contractions are possible as indicated above.

We now use $\quad u(x)_\alpha \bar{u}_\beta(y) = i \int \frac{d^4p}{(2\pi)^4} e^{i(x-y)\cdot p} \frac{(\not{p} + m)_{\alpha\beta}}{p^2 - m^2 + i\varepsilon}$

This leads to (the extra $-$ sign is from the anticommutations to bring u in front of $\bar{u}$)

$$\Pi^{V}_{\mu\nu} = -i^3 \int d^4x \; e^{+iq\cdot x} \int \frac{d^4p}{(2\pi)^4} e^{+ip\cdot x} \int \frac{d^4p'}{(2\pi)^4} e^{-ip'\cdot x}$$
$$\times \; \mathrm{tr} \frac{\gamma_\mu (\not{p}' + m) \gamma_\nu (\not{p} + m)}{(p^2 - m^2 + i\varepsilon)(p'^2 - m^2 + i\varepsilon)}$$

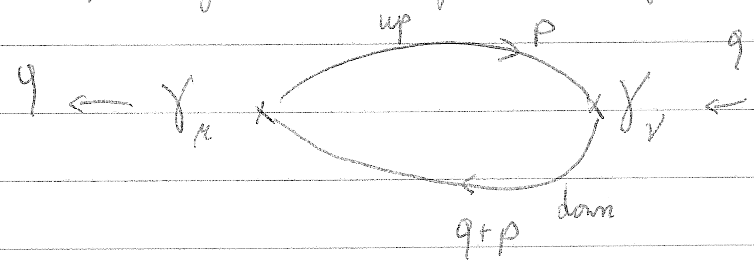
We regulate all infinities by working in noninteger dimensions and now use $d = 4 - 2\varepsilon$ with ε small and
(Notice that this is not the ε in the $i\varepsilon$ in the denominators)

$$\int d^d x \; e^{i q \cdot x} = \delta^d(q) \, (2\pi)^d$$

to obtain

$$\Pi^V_{\mu\nu} = i \int \frac{d^d p}{(2\pi)^d} \; \frac{\mathrm{tr}\, \gamma_\mu (\not{q} + \not{p} + m) \gamma_\nu (\not{p} + m)}{(p^2 - m^2 + i\varepsilon)\,((p+q)^2 - m^2 + i\varepsilon)}$$

corresponding to the Feynman diagram



The d-dimensional γ matrices satisfy

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 g_{\mu\nu}$$

$$g^\mu{}_\mu = d$$

$$\mathrm{tr}(\gamma_\mu \gamma_\nu) = 2^{d/2} g_{\mu\nu} \qquad \text{(but 4 is also an acceptable choice)}$$

$$\mathrm{tr}(\gamma_\mu \gamma_\nu \gamma_\alpha \gamma_\beta) = 2^{d/2} \left(g_{\mu\nu} g_{\alpha\beta} - g_{\mu\alpha} g_{\nu\beta} + g_{\mu\beta} g_{\nu\alpha} \right)$$

and the trace of an odd number of gamma matrices vanishes.

$$\Rightarrow \quad \Pi^{V}_{\mu\nu} = i\, 2^{d/2} \int \frac{d^d p}{(2\pi)^d} \frac{\left[m^2 g_{\mu\nu} + (q+p)_\mu p_\nu - p\cdot(q+p)\, g_{\mu\nu} + p_\mu (q+p)_\nu \right]}{(p^2-m^2)\left((p+q)^2 - m^2\right)}$$

We simplify the three integrals using Feynman's trick:

$$\frac{1}{AB} = \int_0^1 dx \, \frac{1}{\left[xA + (1-x)B \right]^2} \qquad \text{with } B = p^2 - m^2$$

$$\Pi^{V}_{\mu\nu} = i\, 2^{d/2} \int_0^1 dx \int \frac{d^d p}{(2\pi)^d} \frac{[\qquad\qquad]}{\left[(p+xq)^2 - \left(m^2 - x(1-x)q^2\right) \right]^2}$$

Now the integral is a lot easier when we introduce $\tilde{p} = p + xq$ and shift to $\tilde{p}$ as integration variable instead and introduce $M^2 = m^2 - x(1-x)q^2$

$$\Pi^{V}_{\mu\nu} = i\, 2^{d/2} \int_0^1 dx \int \frac{d^d \tilde{p}}{(2\pi)^d} \frac{g_{\mu\nu}\left(m^2 - \tilde{p}^2 - (1-2x)\tilde{p}\cdot q + x(1-x) q^2\right) + C}{\left[\tilde{p}^2 - M^2 + i\varepsilon \right]^2}$$

$$\text{with} \quad C = 2\tilde{p}_\mu \tilde{p}_\nu + (p_\mu q_\nu + p_\nu q_\mu)\, 2(1-2x) - 2 q_\mu q_\nu\, x(1-x)$$

We now use:

$$\int \frac{d^d p}{(2\pi)^d} \frac{p_\mu p_\nu}{(p^2-m^2)^n} = \frac{1}{d} \int \frac{d^d p}{(2\pi)^d} \frac{p^2}{(p^2-m^2)^n}$$

$$\int \frac{d^d p}{(2\pi)^d} \frac{p_\mu}{(p^2-m^2)^n} = 0$$

and $p^2 = (p^2 - M^2) + M^2$

and $\Pi^V_{\mu\nu} = i\, 2^{d/2} \int_0^1 dx \int \frac{d^d\tilde{p}}{(2\pi)^d} \frac{+ g_{\mu\nu}\left(\frac{2}{d}-1\right)}{\tilde{p}^2 - M^2 + i\varepsilon} + \frac{g_{\mu\nu}\left[x(1-x)q^2 + \frac{2}{d}M^2\right] - 2x(1-x)q_\mu q_\nu}{(\tilde{p}^2 - M^2 + i\varepsilon)^2}$

at this level we can use

$$\frac{1}{i}\int \frac{d^dp}{(2\pi)^d} \frac{1}{p^2 - m^2} = \frac{m^2}{16\pi^2}\left(\frac{1}{\varepsilon} - \gamma + \ln(4\pi) + 1 - \ln\frac{m^2}{\mu^2}\right)$$

μ is a scale introduced to make all necessary couplings dimensionless and will always show up correctly in the argument of logarithms.

We also get (by taking $\frac{\partial}{\partial m^2}$ of the above)

$$\frac{1}{i}\int \frac{d^dp}{(2\pi)^d} \frac{1}{(p^2 - m^2)^2} = \frac{1}{16\pi^2}\left(\frac{1}{\varepsilon} - \gamma + \ln(4\pi) - \ln\frac{m^2}{\mu^2}\right)$$

or

$$\Pi^V_{\mu\nu} = \frac{i^2\, 2^{d/2}}{16\pi^2}\int_0^1 dx\, (q_\mu q_\nu - q^2 g_{\mu\nu})\,(-2)x(1-x)\left(\frac{1}{\varepsilon} - \gamma + \ln(4\pi) - \ln\frac{M^2}{\mu^2}\right)$$

notice that the current is transverse as it should be. $(q^\mu \Pi^V_{\mu\nu} = 0)$

We use now $\int_0^1 dx\, x(1-x) = \frac{1}{6}$

$q^2 < 0$: $\ln\, m^2 - x(1-x)q^2 + i\varepsilon \Rightarrow \ln x + \ln(1-x) + \ln(-q^2)$

$q^2 > 0$: $\ln\, m^2 - x(1-x)q^2 - i\varepsilon \Rightarrow \ln x + \ln(1-x) + \ln(q^2) - i\pi$

$\int_0^1 dx\, x(1-x)\ln x = \frac{-5}{36}$ and the $x \leftrightarrow (1-x)$ symmetry

$$\Pi^V_{\mu\nu} = (q_\mu q_\nu - q^2 g_{\mu\nu})\frac{2\cdot 2^{d/2}}{16\pi^2}\left\{\frac{1}{6}\left(\frac{1}{\varepsilon} - \gamma + \ln(4\pi) - \ln(-q^2)\right) + \frac{5}{18}\right\}$$

or $$\Pi^V_{\mu\nu} = \left(q_\mu q_\nu - q^2 g_{\mu\nu}\right) \frac{1}{4\pi^2} \frac{1}{3} \left[\frac{1}{\varepsilon} - \gamma + \ln 4\pi - \ln 2 - \ln\left(\frac{-q^2}{\mu^2}\right) + \frac{5}{3} \right]$$

Now we remove the $\frac{1}{\varepsilon}$ using MS. The divergence comes from the region $x \simeq 0$ where both currents come on top of each other. So we can subtract any finite amount there as well so only the $\ln(-q^2)$ term is physically relevant.

We have 3 possible colours giving an extra overall factor of 3.

Alternatively: using $\hat{L}_M$ only the $\ln(-q^2)$ gives contributions in the QCD sum rules