# About matrices, tensors and various abbreviations

The objects we deal with in this course are rather complicated. We therefore use a simplifying notation where at every step as much as possible of this complexity is hidden. Basically this means that objects that are made out of many elements or components. are written without indices as much as possible. We also have several different types of indices present. The ones that show up in this course are:

- Dirac-Indices $a, b, c$
- Lorentz indices, both upper and lower $\mu, \nu, \rho$
- $SU(2)_L$ indices $i, j, k$
- $SU(3)_c$ indices $\alpha, \beta, \gamma$

Towards the right I have written the type of symbols used in this note to denote a particular type of index. The course contains even more, three-vector indices or various others denoting sums over types of quarks and/or leptons.

An object is called scalar or singlet if it has no index of a particular type of index, a vector if it has one, a matrix if it has two and a tensor if it has two or more.

A vector can also be called a column or row matrix. Examples are $b$ with elements $b_i$:

$$b = (b_1, b_2, \ldots, b_n) \quad b = (b_1\ b_2\ \cdots\ b_n) \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \tag{1}$$

In the first case is a vector, the second a row matrix and the last a column vector.

Objects with two indices are called a matrix or sometimes a tensor and denoted by

$$c = (c_{ij}) = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{pmatrix} \tag{2}$$

which is referred to a an $m \times n$ or an $m$ by $n$ matrix. It has $m$ rows and $n$ columns.

Objects with more than two indices cannot be so simply written. But the elements of vectors and matrices can themselves be matrices and vectors in other indices again. Examples of some objects that show up in the course are:

$$Q_{ai\alpha} \qquad F_{\mu\nu\alpha\beta} \qquad \gamma_{\mu ab} \qquad A_{\mu\alpha\beta} \tag{3}$$

Q is a two element column vector in $SU(2)_L$ indices, a three element column vector in $SU(3)_c$ indices and a four-vector in Dirac indices. $\gamma_{\mu ab}$ is a four-vector in Lorentz indices but a matrix in Dirac indices and similarly for the other quantities.

In general we will also use Einstein notation, i.e. if indices are repeated a summation over them is implied.

For matrices we define a few additional quantities:

- Matrix multiplication: $c = ab$ with $c_{ij} = a_{ik} b_{kj}$ with $k$ summed over. So a product of an $m \times n$ matrix with a $n \times l$ matrix gives a $m \times l$ matrix. E.g.

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

- The trace of a matrix is the sum of its diagonal components. $tr(a) = a_{ii}$. This operation satisfies $tr(ab) = tr(ba)$, called cyclicity.

- The transpose of a matrix $a$ is a matrix $a^T$ with the elements mirrored. $(a^T)_{ij} = a_{ji}$.

- The hermitian conjugate of a matrix $a$ is a matrix $a^\dagger$ with the elements mirrored and complex conjugated. For vectors this changes a column vector to a row vector. $(a^\dagger)_{ij} = (a_{ji})^*$

- The determinant of matrix is somewhat more difficult to define but for a 2 by 2 matrix it is $\det(a) = a_{11}a_{22} - a_{21}a_{12}$ and for a 3 by 3 matrix $\det(b) = b_{11}b_{22}b_{33} + b_{12}b_{23}b_{31} + b_{13}b_{32}b_{21} - b_{13}b_{22}b_{31} - b_{11}b_{32}b_{23} - b_{21}b_{12}b_{33}$. It satisfies $\det(ab) = \det(a)\det(b)$.

A few other things which show up are the product of Lorentz vectors

$$p \cdot q = p_\mu q^\mu = p^\mu q_\mu = p^0 q^0 - p^1 q^1 - p^2 q^2 - p^3 q^3 \tag{4}$$

Lorentz indices are brought up by the matrix $g^{\mu\nu}$ and down by $g_{\mu\nu}$ which both are in matrix notation

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \text{diag}(1, -1, -1, -1) \tag{5}$$

For Dirac fermions (see chapter 5) we also introduce $\overline{\psi} = \psi^\dagger \gamma^0$ and the slash notation $\not{p} = p^\mu \gamma_\mu$.

So as an example we have

$$\overline{Q} \not{A} Q = (Q_{ai\alpha})^* \, \gamma^0_{ab} \gamma_{\mu bc} A^\mu_{\alpha\beta} Q_{ci\beta} \tag{6}$$