Warm-up: Fermat's principle

Pierre de Fermat
1601-1665

L1:1
Taylor:
215-225

One formulation of Fermat's principle is to say that light follows paths which minimize time. We will use an alternative formulation:
Light follows a path for which all nearby paths would have taken as long time.

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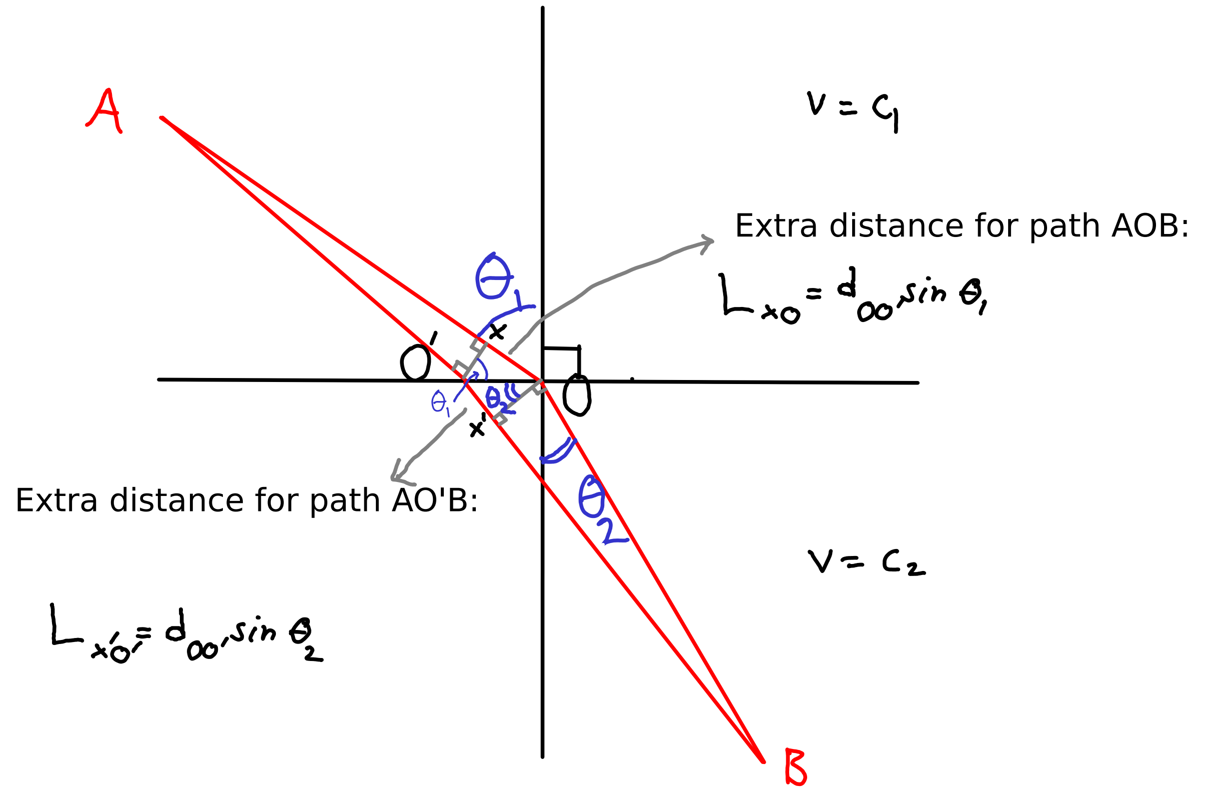


Compare the paths AOB and AO'B for O' infinitesimally close to O.

On path AO'B you gain time: $\dfrac{L_{xO}}{c_1}$

but you lose time: $\dfrac{L_{x'O}}{c_2}$

For the sought path we thus have:

$$\frac{L_{xO}}{c_1} = \frac{L_{x'O}}{c_2}$$

$$\Rightarrow \quad d_{OO'} \frac{\sin\theta_1}{c_1} = d_{OO'} \frac{\sin\theta_2}{c_2}$$

$$\Rightarrow \quad \frac{\sin\theta_1}{\sin\theta_2} = \frac{c_1}{c_2}.$$

Snell's law

Willebrord Snellius
(1580–1626)

Question: Why does light follow straight paths?

Light follows paths which extremize the time and if c is the same

the distance will also be extremized => Light follows straight paths

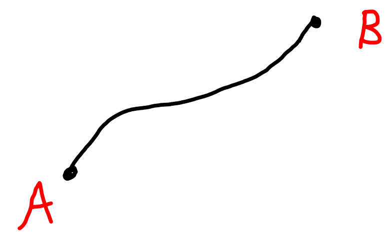
(We will prove this!)

Calculus of variations

Fermat's principle says that ligt follows paths which extremizes the time.

Given two points A and B there are many paths connecting them

$$\Gamma : y = y(x)$$



How do we find the shortest path?

More generally: How do we extremize an integral? (Such as the integral of the distance from A to B)

The length of a path is given by

$$S_{AB} = \int_A^B ds = \int_A^B \sqrt{dx^2 + dy^2} = \int_A^B \sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_A^B \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_A^B \sqrt{1 + y'^2}\, dx$$

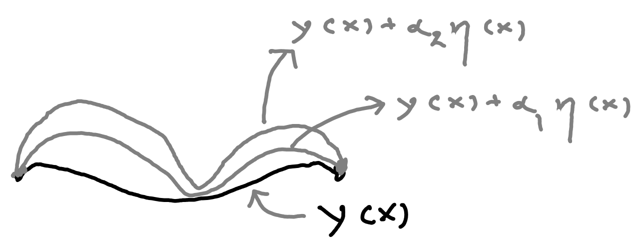
We want to consider small variations and require the path length (or integral) to remain the same.

Consider:

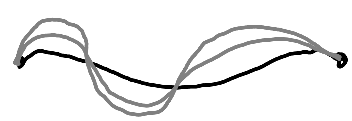
(Think about minimizing a function: To find the minimum we put the derivative to 0. Similarily we here make the lowest order variation vanish.)

$$\Gamma(\alpha) = y(x, \alpha) = y(x) + \underbrace{\alpha\,\eta(x)}$$

small deformation with $\eta(A) = \eta(B) = 0$



same η different α



same y, α_1 α_2
but new η

We want the distance to $\underbrace{\text{be invariant}}_{\text{be the same}}$ under small deformations

$$\Rightarrow S_{AB}(\alpha) = S_{AB}(0) + \left.\frac{\partial S_{AB}}{\partial \alpha}\right|_{\alpha=0} \cdot \alpha + \frac{1}{2} \left.\frac{\partial^2 S_{AB}}{\partial \alpha^2}\right|_{\alpha=0} \cdot \alpha^2 + \cdots$$

so S_{AB} constant

$$\Rightarrow 0 = \left.\frac{\partial S_{AB}}{\partial \alpha}\right|_{\alpha=0} = \frac{\partial}{\partial \alpha} \underbrace{\int_A^B \sqrt{1 + \left(\frac{\partial y(x,\alpha)}{\partial x}\right)^2}\, dx}_{S(\alpha)} \Bigg|_{\alpha=0}$$

Using implicit evaluation in $\alpha = 0$ we see that this is a condition of type:

$$0 = \frac{\partial S(\alpha)}{\partial \alpha} = \frac{\partial}{\partial \alpha} \int_A^B f(y'(x,\alpha))\, dx$$

Consider now a more general case:

$$0 = \frac{\partial}{\partial \alpha} \int_A^B f(y, y', x)\, dx = \frac{\partial}{\partial \alpha} \int_A^B f\big(y(x,\alpha), y'(x,\alpha), x\big)\, dx$$

y and y' depend on x and α

f depends on x and y and y'

In order for the variation to vanish we have

$$0 \overset{!}{=} \frac{\partial}{\partial \alpha} \int_A^B f\left(y(x,\alpha), y'(x,\alpha), x\right) dx$$

chain rule

$$= \int_A^B \Bigg[\overbrace{\frac{\partial f}{\partial y}\left(\frac{\partial}{\partial \alpha} y(x,\alpha)\right)}^{\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha}} + \overbrace{\frac{\partial f}{\partial y'}\left(\frac{\partial}{\partial \alpha} y'(x,\alpha)\right)}^{\frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha}} + \underbrace{\frac{\partial f}{\partial x}\frac{\partial x}{\partial \alpha}}_{0} \Bigg] dx$$

change order of differentiation

so x-dependence doesn't matter

change order of partial derivatives

$$= \int_A^B \Bigg[\underbrace{\qquad\qquad}_{\eta(x)} + \frac{\partial f}{\partial y'} \underbrace{\left(\frac{\partial}{\partial x} \underbrace{\frac{\partial}{\partial \alpha} y(x,\alpha)}_{\eta(x)}\right)}_{\frac{d\eta}{dx}} \Bigg] dx$$

$$y(x,\alpha) = y(x) + \alpha\eta(x)$$

ok with full derivative as only x-dep.

$$= \int_A^B \left[\frac{\partial f}{\partial y}\eta(x) + \frac{\partial f}{\partial y'}\frac{d\eta(x)}{dx} \right] dx$$

use integration by parts for 2nd term

$$= \int_A^B \left[\qquad - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta(x) \right] dx + \underbrace{\frac{\partial f}{\partial y'}\eta(x)\Big|_A^B}_{0 \text{ as } \eta(A) = \eta(B) = 0}$$

$$= \int_A^B \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \right] \eta(x)\, dx \overset{!}{=} 0$$

But this should be valid for ALL $\eta(x)$ i.e. all variations of y

$$\Rightarrow \boxed{\frac{\partial f}{\partial y} = \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)}$$

very important result!

Euler-Lagrange equation

Note: Partial derivatives w.r.t. y and y' and total derivative w.r.t. x

Taylor:
216-217

Example:

For the case of the shortest path in the Euclidean plane we have

$$f(y, y') = f(y') = \sqrt{1+y'^2}$$ (See p. 3)

so the Euler-Lagrange equation gives

$$\underbrace{\frac{\partial f}{\partial y}}_{0} = \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$$

Note: For $\frac{\partial f}{\partial y}$ y' is considered independent of y since variations of y' are taken care of separately.

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{2}\frac{2y'}{\sqrt{1+y'^2}}\right) = 0$$

$$\Rightarrow \frac{y'}{\sqrt{1+y'^2}} = \text{const}$$

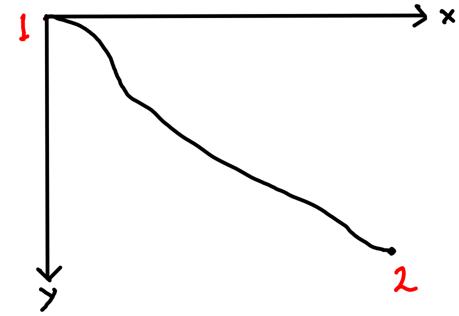
$$\Rightarrow y' \quad \text{constant}$$

$$\Rightarrow y = kx + m \quad \Rightarrow \quad \text{a straight line!}$$

So if the distance is given by $dx^2 + dy^2$ the shortest paths are straight lines.

The Brachistochrone problem

Let a particle move in a gravitational field along a path between two points 1 and 2.



What path will give the shortest time, assuming $v_1 = 0$ and $E_{pot} \to E_{kin}$?

Choose to let x depend on y
x=x(y)

We want to minimize

$$t_{12} = \int_1^2 \frac{ds}{v} = \int_1^2 \frac{\sqrt{(dx)^2 + (dy)^2}}{v} = \int_1^2 \frac{\sqrt{\left(\frac{dx}{dy}\right)^2 + 1}}{v} dy = \int_1^2 \frac{\sqrt{x'^2 + 1}}{v} dy$$

As E_{tot} is conserved we have

$$\frac{1}{2} \cancel{m} v^2 = \cancel{m} g y \Rightarrow v = \sqrt{2gy}$$

$$\Rightarrow t_{12} = \frac{1}{\sqrt{2g}} \int_1^2 \underbrace{\frac{\sqrt{x'^2 + 1}}{\sqrt{y}}}_{f(x,\, x',\, y)} dy$$

depends on y corresponding to x

this is the function to minimize

To minimize the time we use the Euler-Lagrange equation

Note that our f is independent of x.

Comment: x'(y) is considered to be independent of x(y). This is the case since the variations in x' already have been considered (in the x' differentiation).

$$\frac{\partial f}{\partial x} = \frac{d}{dy}\left(\frac{\partial f}{\partial x'}\right)$$

$$0 = \frac{d}{dy}\left(\frac{1}{\sqrt{y}}\frac{x'}{\sqrt{x'^2+1}}\right)$$

For convenience

$$\Rightarrow \frac{1}{y}\frac{x'^2}{x'^2+1} = \text{const} = \frac{1}{2a}$$

$$\Rightarrow x'^2 = \frac{y}{2a}(1+x'^2)$$

$$\Rightarrow x'^2\left(1-\frac{y}{2a}\right) = \frac{y}{2a}$$

$$\Rightarrow x'^2 = \frac{y}{2a-y}$$

$$\Rightarrow x = {+ \atop (-)}\int \sqrt{\frac{y}{2a-y}}\, dy$$

This integral can be evaluated with the substitution:

$$y = a(1-\cos\theta) \Rightarrow dy = a\sin\theta\, d\theta$$

This ansatz only works if y<=a. We will see below that this is the case.

$$x = \int \underbrace{\sqrt{\frac{\cancel{a}(1-\cos\theta)}{\cancel{a}(1+\cos\theta)}}}_{\sqrt{\frac{(1-\cos\theta)^2}{\cancel{1-\cos\theta^2}}}}\; a\,\cancel{\sin\theta}\, d\theta$$

$0 \le \theta \le \pi$

$$= a\int(1-\cos\theta)\, d\theta$$

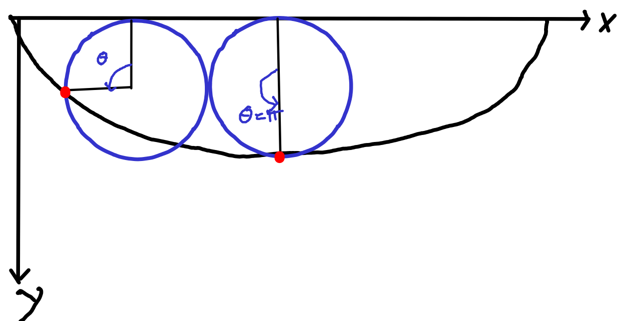
$$= a(\theta - \sin\theta) + \text{const}$$

So we have:

$$\begin{pmatrix} x \\ y \end{pmatrix} = a \begin{pmatrix} \theta \\ 1 \end{pmatrix} - a \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}$$

describes the movement of the center of the circular motion

describes a circular motion



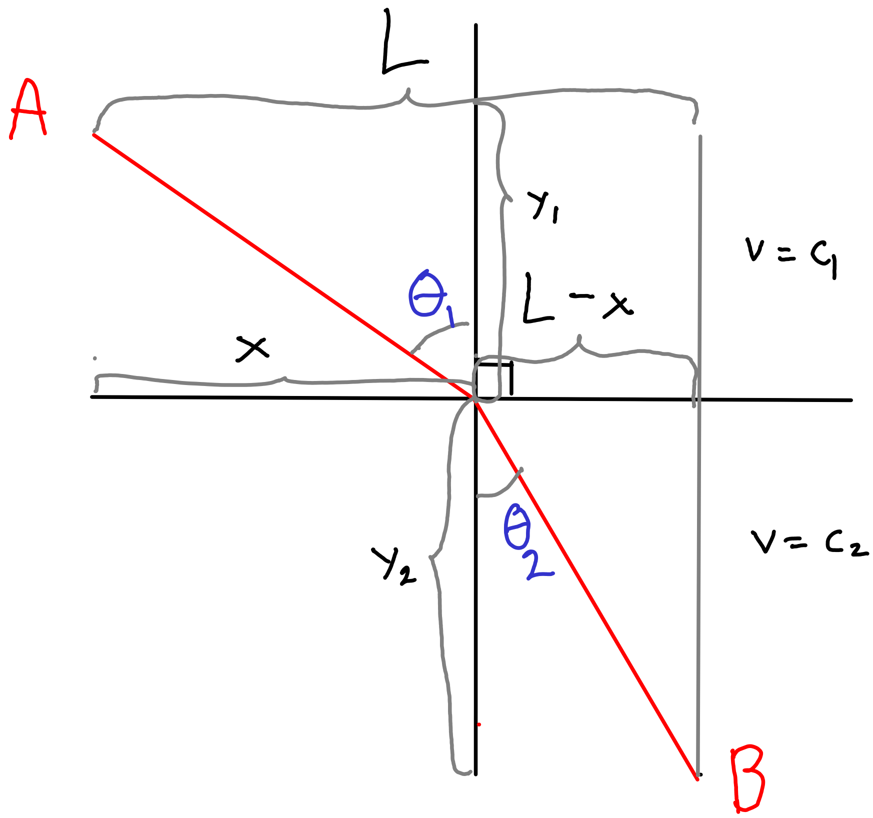
The solution is a cycloid.

Extra: Fermat's principle, shortest time

Pierre de Fermat
1601-1665

(not in Taylor)

Formulation 1: Light follows the path that takes shortest time.



$$t_{AB} = t_1 + t_2 = \frac{\sqrt{x^2 + y_1^2}}{c_1} + \frac{\sqrt{(L-x)^2 + y_2^2}}{c_2}$$

Find minimum: $$\frac{dt_{AB}}{dx} = 0$$

$$\Rightarrow \frac{1}{\cancel{2}} \frac{\cancel{2}x}{\sqrt{x^2 + y_1^2}} \frac{1}{c_1} - \frac{1}{\cancel{2}} \frac{\cancel{2}(L-x)}{\sqrt{(L-x)^2 + y_2^2}} \frac{1}{c_2} = 0$$

$$\Rightarrow \frac{1}{c_1} \sin\theta_1 = \frac{1}{c_2} \sin\theta_2$$

$$\Rightarrow \frac{\sin\theta_1}{\sin\theta_2} = \frac{c_1}{c_2} = \text{const} = \frac{n_2}{n_1}$$ Snell's law