Conservation of total momentum

Taylor: 268-269

We will here prove that conservation of total momentum is a consequence of translational invariance.

Consider N particles with positions given by $\bar{r}_1, \bar{r}_2, \dots \bar{r}_N$.

If we translate the whole system with some distance $\bar{\varepsilon}$ we get

$$\bar{r}_1 + \bar{\varepsilon},\ \bar{r}_2 + \bar{\varepsilon},\ \dots\ \bar{r}_N + \bar{\varepsilon}$$

The potential energy must be unaffected by the translation.

(We move the "whole world", no relative position change.)

$$V(\bar{r}_1, \bar{r}_2, \cdots \bar{r}_N) = V(\bar{r}_1 + \bar{\varepsilon}, \bar{r}_2 + \bar{\varepsilon}, \cdots \bar{r}_N + \bar{\varepsilon})$$

$$\Rightarrow \quad \delta V = 0$$

Also, clearly all velocities are independent of the translation $(\dot{\bar{\varepsilon}} = 0)$.

$$\Rightarrow \quad \delta T = 0$$

$$\Rightarrow \quad \delta L = \delta T - \delta V = 0$$

Let's consider $\bar{\varepsilon}$ to be in the x-direction, then

$$0 = \delta L = \varepsilon \frac{\partial L}{\partial x_1} + \cdots + \varepsilon \frac{\partial L}{\partial x_N}$$

$$\Rightarrow \quad \sum_{\alpha=1}^{N} \frac{\partial L}{\partial x_\alpha} = 0$$

Using Lagrange's equation, we can rewrite each derivative as

$$\frac{\partial L}{\partial x_\alpha} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_\alpha}\right) = \frac{d}{dt}(m_\alpha \dot{x}_\alpha) = \dot{p}_{\alpha x}$$

$$\Rightarrow \quad \sum_{\alpha=1}^{N} \dot{p}_{\alpha x} = 0$$

I.e., total momentum is conserved!

(Clearly there is nothing special about the x-direction.)

Generalized coordinates, velocities, momenta and forces

Taylor: 241-242

With $L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z)$

we have $\frac{\partial L}{\partial \dot{x}} = m\dot{x} = p_x$

i.e., differentiating w.r.t. $\dot{x}$ gives us the momentum in x -direction.

generalized velocity

In analogy we define the generalized momentum $p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$

and we say that p_i and q_i are conjugated variables.

We also define the generalized force $Q_i = \frac{\partial L}{\partial q_i}$

Note: Def. of gen. force from Taylor 7.15 differs from def in HUB I.9.17.

such that

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \Leftrightarrow Q_i = \frac{d}{dt} p_i$$

generalized force — rate of change of generalized momentum

Note: q_i (generalized coordinate) need not have dimension length

$\dot{q}_i$ (generalized velocity) velocity

p_i (generalized momentum) momentum

Q_i (generalized force) force

Example: For a system which rotates around the z-axis we have for V=0

$$L = E_{k,\varphi} = \frac{1}{2} I_z \dot{\varphi}^2$$

s.t. $\frac{\partial L}{\partial \dot{\varphi}} = I_z \dot{\varphi} = L_z$

The angular momentum is thus conjugated variable to φ

length	x	$\leftrightarrow$	φ	dimensionless
length/time	v	$\leftrightarrow$	$\dot{\varphi}$	1/time
mass*length/time	p	$\leftrightarrow$	L_z	mass*(length)^2/time
	F	$\leftrightarrow$	$\frac{\partial L}{\partial \varphi}$ (torque)	energy

Taylor: 266-267

Cyclic coordinates = ignorable coordinates and Noether's theorem

We have defined the generalized momentum $p_i = \frac{\partial L}{\partial \dot{q}_i}$

From EL we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$

i.e.

$$\frac{d}{dt}(p_i) = \frac{\partial L}{\partial q_i}$$

but this means that if L is independent of q_i the corresponding conjugated momentum is conserved!!!

A generalized coordinate that does not explicitly enter the Lagrangian is called a cyclic coordinate and the corresponding conserved quantity is called a constant of motion.

That we have a constant of motion when L is independent of the corresponding conjugated coordinate is a special case of Noether's theorem which states:

If a system has a continuous symmetry, there is a corresponding conserved quantity.

(Emmy Noether 1882 – 1935)

Ex 1: Assume that L does not depend explicitly on x and V not on $\dot{x}$,

then: $$\frac{\partial L}{\partial x} = 0 \Rightarrow 0 = \frac{d}{dt}\underbrace{\left(\frac{\partial L}{\partial \dot{x}}\right)}_{\frac{\partial}{\partial \dot{x}}\left(\frac{m\dot{x}^2}{2}\right)} = \frac{d}{dt} m\dot{x} = \frac{d}{dt} p_x$$

=> momentum in x-direction is conserved!

Ex 2: Assume that a particle moves in a central potential, s.t.

$$L = E_k - E_p = \frac{1}{2} m \underbrace{\left(\dot{r}^2 + r^2\dot{\varphi}^2\right)}_{\text{no } \varphi \text{ only } \dot{\varphi}} - V(r)$$

In this case L does not depend explicitly on φ and we have a constant of motion

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m r^2 \dot{\varphi}$$

the conjugated momentum to , the angular momentum, is conserved!

Conservation of Energy

We have seen that cyclic coordinates correspond to conserved quantities.

What about energy? $\longleftrightarrow$ time? Time is special as we extremize a time integral in analytical mechanics

number of generalized coordinates

$$\text{Study}\quad \frac{d}{dt}L(\bar{q},\dot{\bar{q}},t) = \overbrace{\sum_{i=1}^{n}\left[\underbrace{\frac{\partial L}{\partial q_i}}_{\text{EL: } \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i\right]}^{\text{implicit time dependence}} + \overbrace{\frac{\partial L}{\partial t}}^{\text{explicit}}$$

$$= \sum_{i=1}^{n}\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right] + \frac{\partial L}{\partial t}$$

$$= \frac{d}{dt}\left[\sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right] + \frac{\partial L}{\partial t}$$

i.e.

$$\frac{d}{dt}\underbrace{\left[\sum_{i=1}^{n}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right) - L\right]}_{H} = -\frac{\partial L}{\partial t}$$

s.t. the quantity $H(q_1\cdots q_n, \dot{q}_1\cdots\dot{q}_n, t) = \sum_{i=1}^{n}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right) - L$

is conserved if $\frac{\partial L}{\partial t} = 0$ i.e. if L doesn't depend <u>explicitly</u> on time.

H is called <u>Hamiltonian</u> and is thus conserved under time translations

i.e., doesn't change with time, if $\frac{\partial L}{\partial t} = 0$.

Note that H can be written

$$H = \sum_{i=1}^{n} \underbrace{\frac{\partial L}{\partial \dot{q}_i}}_{p_i} \dot{q}_i - L$$

If L does depend explicitly on t
H may change with time

see $\dot{q}_i$ as depending on $p_1 \cdots p_n$

$$= \sum_{i=1}^{n} p_i \dot{q}_i(p_1, \ldots, p_n) - L = H(q_1 \cdots q_n, p_1 \cdots p_n, t)$$

The Hamiltonian can be seen as a function of

- generalized momenta (p_i)
- generalized coordinates (q_i)

Hamilton's formulation of mechanics and quantum mechanics

rather than

- generalized velocities $(\dot{q}_i)$
- generalized coordinates (q_i)

What is H?

Assume that we have N particles and that V only depends on generalized coordinates

$$L = \sum_{\alpha=1}^{N} \frac{1}{2} m_\alpha \dot{\bar{r}}_\alpha^2 - V(\bar{r}_1 \cdots \bar{r}_N) = T - V$$

If we have natural generalized coordinates $\bar{r}_\alpha = \bar{r}_\alpha \underbrace{(q_1 \cdots q_n)}_{\text{no } t \text{ for natural coordinates}} \ \forall \alpha$

(Natural coordinates, Taylor p. 249)

$$\Rightarrow \dot{\bar{r}}_\alpha = \sum_{i=1}^{n} \frac{\partial \bar{r}_\alpha}{\partial q_i} \dot{q}_i$$

$$\Rightarrow \dot{\bar{r}}_\alpha^2 = \left(\sum_{i=1}^{n} \frac{\partial \bar{r}_\alpha}{\partial q_i} \dot{q}_i \right) \cdot \left(\sum_{j=1}^{n} \frac{\partial \bar{r}_\alpha}{\partial q_j} \dot{q}_j \right)$$

T is thus a quadratic form in generalized velocities

$$T = \sum_\alpha \frac{m \dot{\bar{r}}_\alpha^2}{2} = \sum_{i,j} \frac{1}{2} A_{ij} \dot{q}_i \dot{q}_j$$

where
$$\begin{cases} A_{ij} = A_{ij}(q_1 \cdots q_n) \\ A_{ij} = A_{ji} \quad \text{(A can always be chosen symmetric)} \end{cases}$$

giving the derivatives

(change name of summation index in 2nd term)

$$\frac{\partial T}{\partial \dot{q}_k} = \sum_{i,j} \frac{1}{2} A_{ij} \underbrace{\frac{\partial \dot{q}_i}{\partial \dot{q}_k}}_{\delta_{ik}} \dot{q}_j + \sum_{i,j} \frac{1}{2} A_{ij} \dot{q}_i \underbrace{\frac{\partial \dot{q}_j}{\partial \dot{q}_k}}_{\delta_{jk}} = \sum_j \dot{q}_j A_{kj}$$

The first term in H can be written

$$\sum_k \frac{\partial L}{\partial \dot{q}_k} \dot{q}_k = \sum_k \frac{\partial T}{\partial \dot{q}_k} \dot{q}_k = \sum_k \left(\sum_j \dot{q}_j A_{kj} \right) \dot{q}_k = \sum_{k,j} \dot{q}_k \dot{q}_j A_{k,j} = 2T$$

Inserting into H we find

$$H = \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L$$

$$= 2T - (T - V) = T + V = E \ !$$

So, if the coordinate transformations from Cartesian to generalized coordinates are time independent and V does not depend on $\dot{q}_i$ the Hamiltonian is just the energy.

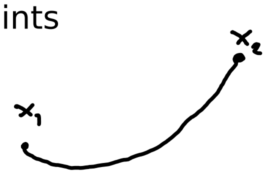
Especially in Cartesian coordinates E=H is conserved if L does not depend explicitly on time.

Constraints and Lagrange multipliers, the hanging chain

(Not in Taylor)

Consider a chain with length l that hangs between two points

at distance smaller than l.



The chain will hang s.t. the potential energy is minimized,

What shape does that imply?

Question: Can we solve this by minimizing

$$E_p = \int dm\, g\, y = \int \mu\, ds\, g\, y = \mu g \int y\, ds \quad ?$$

assume constant density μ

Answer: No! We have a constraint, the length must be l , i.e.

$$\int_{x_1}^{x_2} \sqrt{1+y'^2}\, dx \overset{!}{=} l \, .$$

The length of the chain may not be varied => for viable solutions we have:

$$0 = \delta l = \delta\left(\int_{x_1}^{x_2} \sqrt{1+y'^2}\, dx \right)$$

We have two non-independent constraints on the minimizig shape.

Since $\delta E_p = 0$ and $\delta l = 0$ each linear combination must vanish

$$0 = \frac{1}{\mu g}\, \delta E_p - \lambda\, \delta l = \int_{x_1}^{x_2} \Big(\underbrace{y\sqrt{1+y'^2} - \lambda\sqrt{1+y'^2}}_{f(y,\, y')} \Big)\, dx$$

Lagrange multiplier - some constant

We may as well pick the constants such that the calculations are simplified.

EL $\Rightarrow$

$$0 = \frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} \qquad f(y,y') = y\sqrt{1+y'^2} - \lambda\sqrt{1+y'^2}$$

consider
y and y' indep

$$= (1+y'^2)^{1/2} - \frac{d}{dx}\left((y-\lambda)\frac{1}{\cancel{2}}\frac{\cancel{2}y'}{(1+y'^2)^{1/2}} \right)$$

don't forget
inner derivative

$$= (\quad)^{1/2} - \left[\frac{y'^2}{(\quad)^{1/2}} + \frac{(y-\lambda)y''}{(\quad)^{1/2}} + (y-\lambda)y'\left(-\frac{1}{\cancel{2}}\right)\frac{\cancel{2}y'y''}{(\quad)^{3/2}} \right]$$

collect in front
of $\frac{1}{(1+y'^2)^{3/2}}$

$$= \frac{1}{(1+y'^2)^{3/2}}\Big[\underbrace{(1+y'^2)^2}_{1+2y'^2+\cancel{y'^4}} - y'^2(1+\cancel{y'^2}) - (y-\lambda)(1+\cancel{y'^2})y'' + \cancel{(y-\lambda)y'^2y''} \Big]$$

$$= \frac{1}{(\quad)^{3/2}}\Big[1 + y'^2\underbrace{(2-1)}_{1} - (y-\lambda)y'' \Big]$$

$$\Rightarrow\quad 1 + y'^2 - y''(y-\lambda) = 0$$

$$\Rightarrow\quad 0 = \frac{1+y'^2}{y-\lambda} - y'' \qquad \cdot\left|\ \frac{y'}{1+y'^2}\right.$$

$$\Rightarrow\quad 0 = \frac{y'}{y-\lambda} - \frac{y''y'}{1+y'^2}$$

$$= \frac{d}{dx}\left[\ln(y-\lambda) - \frac{1}{2}\ln(1+y'^2) \right]$$

$\ln(a)+\ln(b)$
$=\ln(ab)$

$$= \frac{d}{dx}\left[\ln\left(\frac{y-\lambda}{(1+y'^2)^{1/2}} \right) \right]$$

$$\Rightarrow\quad \frac{y-\lambda}{(1+y'^2)^{1/2}} = \text{const} = a$$

$$\Rightarrow\quad y'^2 = \left(\frac{y-\lambda}{a}\right)^2 - 1$$

$$\Rightarrow\quad \frac{dy}{dx} = \pm\sqrt{\left(\frac{y-\lambda}{a}\right)^2 - 1}$$

$$\Rightarrow \quad dx = \pm \frac{dy}{\sqrt{\left(\frac{y-\lambda}{a}\right)^2 - 1}}$$

Let $\quad y = \lambda + a\cosh z \quad \Rightarrow \quad \frac{dy}{dz} = a\sinh z$

$\sinh(z) = \frac{e^z - e^{-z}}{2}$

$\cosh(z) = \frac{e^z + e^{-z}}{2}$

$\cosh^2 z - \sinh^2 z = 1$

$\Rightarrow \sinh^2 z = \cosh^2 - 1$

$$\Rightarrow dx = \pm \frac{a\,\cancel{\sinh z}\,dz}{\underbrace{\sqrt{\underbrace{\left(\frac{\cancel{\lambda} + a\cosh z - \cancel{\lambda}}{a}\right)^2}_{\cosh z^2} - 1}}_{\cancel{\sinh}^2 z}}$$

$$= \pm a\,dz$$

$$\Rightarrow \quad x = \pm a z + b$$

$b \hookrightarrow$ constant of integration

For y we have

$$y = \lambda + a\cosh z = \lambda + a\cosh\Big(\overbrace{\pm\frac{x-b}{a}}^{z}\Big)$$

cosh symmetric => sign irrelevant

The constants a, b, λ are determined s.t. the endpoints are correctly placed and s.t. the total length is l.

The solution is a catenary.