Kepler's laws

Taylor:
293-315

Consider two celestial bodies which affect each other only through gravity

$$L = E_k - E_p = \frac{1}{2} m_1 \dot{\bar{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\bar{r}}_2^{\,2} - - G \frac{m_1 m_2}{|\bar{r}_1 - \bar{r}_2|}$$

2 bodies
3 d.o.f. each
=> 6 d.o.f.

Note that E_p only depends on the distance and introduce

$$\begin{cases} \bar{r} = \bar{r}_1 - \bar{r}_2 \\ \bar{r}_{cM} = \dfrac{m_1 \bar{r}_1 + m_2 \bar{r}_2}{\underbrace{m_1 + m_2}_{M}} = \dfrac{m_1 \bar{r}_1 + m_2 \bar{r}_2}{M} \end{cases}$$

M = total mass

Eliminate $\bar{r}_2$ and solve for $\bar{r}_1$

$$\bar{r} + \frac{M}{m_2} \bar{r}_{cM}$$

$$= \bar{r}_1 - \cancel{\bar{r}_2} + \frac{\cancel{M}}{m_2} \left(\frac{m_1 \bar{r}_1 + m_2 \cancel{\bar{r}_2}}{\cancel{M}} \right)$$

$$= \bar{r}_1 + \frac{m_1 \bar{r}_1}{m_2}$$

$$\overset{* m_2}{\Rightarrow} \quad m_2 \bar{r} + M \bar{r}_{cM} = (m_2 + m_1) \bar{r}_1 = M \bar{r}_1$$

$$\Rightarrow \bar{r}_1 = \bar{r}_{cM} + \frac{m_2}{M} \bar{r}$$

In the same way

$$\bar{r}_2 = \bar{r}_{cM} - \frac{m_1}{M} \bar{r}$$

Expressed in $\bar{r}$ and $\bar{r}_{CM}$ we have

$$L = \frac{1}{2} m_1 \dot{\bar{r}}_1^2 + \frac{1}{2} m_2 \dot{\bar{r}}_2^2 + G\frac{m_1 m_2}{|\bar{r}_1 - \bar{r}_2|}$$

$$L = \frac{1}{2} m_1 \left(\dot{\bar{r}}_{CM} + \frac{m_2}{M}\dot{\bar{r}}\right)^2 + \frac{1}{2} m_2 \left(\dot{\bar{r}}_{CM} - \frac{m_1}{M}\dot{\bar{r}}\right)^2 + \frac{G m_1 m_2}{|\bar{r}|}$$

collect in front of $\dot{\bar{r}}^2$ and $\dot{\bar{r}}^2_{CM}$

$$= \dot{\bar{r}}_{CM}^2 \frac{1}{2}(m_1 + m_2) + \dot{\bar{r}}^2 \frac{1}{2} \overbrace{\left(\frac{m_1 m_2^2 + m_2 m_1^2}{(m_1+m_2)^{\cancel{2}}}\right)}^{m_1 m_2 (\cancel{m_1 + m_2})} + 0 + \frac{G m_1 m_2}{|\bar{r}|}$$

$$= \underbrace{\underbrace{\dot{\bar{r}}_{CM}^2 \frac{1}{2} M}_{\text{CM kinetic energy}}}_{L_{CM}} + \underbrace{\underbrace{\dot{\bar{r}}^2 \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2}}_{\text{kinetic energy from relative motion}} + \underbrace{\frac{G m_1 m_2}{|\bar{r}|}}_{\text{- potential energy}}}_{L_{rel}}$$

Here $\bar{r}_{CM}$ is a cyclic coordinate as L does not depend (explicitly) on $\bar{r}_{CM}$ (only $\dot{\bar{r}}_{CM}$)

=> conserved quantities $\frac{\partial L}{\partial \dot{\bar{r}}_{CM}} = \dot{\bar{r}}_{CM} M$ (All components in one equation.)

Total momentum is thus conserved, the center of mass motion is trivial.

We therefore study L in the CM-system

$$L = \frac{1}{2}\mu \dot{\bar{r}}^2 + \underbrace{\frac{G \mu M}{|\bar{r}|}}_{-V}$$

$\mu \rightarrow$ reduced mass $= \frac{m_1 m_2}{m_1 + m_2}$

[Now only 3 d.o.f.]

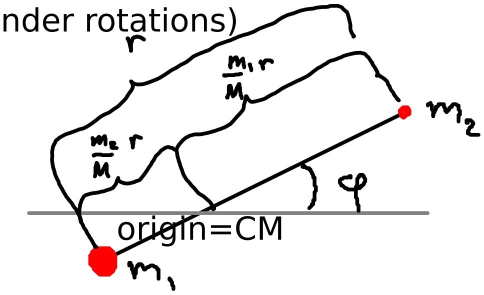
In $\bar{r}$ we have the Lagrange equation $\mu \ddot{\bar{r}} = -\nabla V(\bar{r})$

as if we had one particle with mass μ moving in the potential V

Note that L is spherically symmetric (= invariant under rotations)

=> We can remove one more d.o.f.

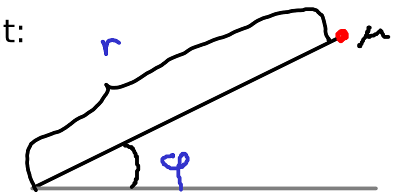
by using $r \equiv |\bar{r}|$ and φ .



What is $\dot{\bar{r}}^2$ in r and φ?

- speed in r-direction gives $\dot{r}^2$
- speed in φ-direction gives $(r\dot{\varphi})^2$

Equivalent:



and motions where only r or only φ change are orthogonal

$$L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) + \frac{G\mu M}{r}$$

In total 2 d.o.f.

no φ dependence

Here φ is a cyclic coordinate

$$\Rightarrow P_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = \mu r^2 \dot{\varphi} \quad = \text{const}$$

generalized momentum

One can prove that P_φ is the angular momentum.

Proof: The total angular momentum is:

$$\bar{r}_1 \times \bar{p}_1 + \bar{r}_2 \times \bar{p}_2$$

$$= m_1 \bar{r}_1 \times \dot{\bar{r}}_1 + m_2 \bar{r}_2 \times \dot{\bar{r}}_2$$

In CM $\bar{r}_1 = \frac{m_2 \bar{r}}{M}$, $\bar{r}_2 = -\frac{m_1 \bar{r}}{M}$

$$= m_1 \left(\frac{m_2}{M}\right)^2 \bar{r} \times \dot{\bar{r}} + \left(\frac{m_1}{M}\right)^2 m_2 \bar{r} \times \dot{\bar{r}}$$

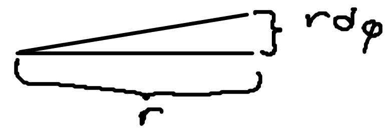
$$= \frac{m_1 m_2}{M^{\cancel{2}}} \cancel{(m_2 + m_1)}\, \bar{r} \times \dot{\bar{r}}$$

$$= \mu\, \bar{r} \times \dot{\bar{r}}$$

with magnitude

$$\mu r^2 \dot{\varphi} = P_\varphi \qquad \underline{\text{ok}}$$

Multiply p_φ with $\frac{dt}{2\mu}$



$$p_\varphi \frac{dt}{2\cancel{\mu}} = \cancel{\mu} r^2 \overbrace{\frac{d\varphi}{dt}}^{\dot\varphi} \frac{dt}{2\cancel{\mu}} = \frac{r}{2}\, r\, d\varphi \approx d(\text{Area}) \doteq \text{(new) const}$$

$\Rightarrow$ Radius vector traverses equal area in equal time, Kepler's 2nd law.

(Johannes Kepler, 1571–1630)

Remark: Kepler's 2nd law is a consequence of p_φ =const, i.e., that we have a central force (it does not depend on the 1/r- dependence)

Kepler's 1st law: The orbit of every planet is an ellipse with the sun at one of the two focal points.

Kepler's 3rd law: $\frac{(\text{orbital period})^2}{r^3}$ same for all planets.

For r, Lagrange's equation gives us

$$\frac{\partial L}{\partial r} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) \qquad L = \frac{1}{2}\mu\left(\dot r^2 + r^2\dot\varphi^2\right) + \frac{G\mu M}{r}$$

$$\Rightarrow \underbrace{\cancel{\mu} r\dot\varphi^2}_{\text{centrifugal force}} - \frac{G\cancel{\mu}M}{r^2} = \cancel{\mu}\ddot r$$

We know that $\mu r^2 \dot\varphi$ = const $\equiv l \;\Rightarrow\; \dot\varphi = \frac{l}{\mu r^2}$

$$\Rightarrow \quad r\left(\frac{l}{\mu r^2}\right)^2 - \frac{GM}{r^2} = \ddot r \qquad (r)$$

Introduce $u(\varphi) = \frac{1}{r(\varphi)}$

We assume that r can be seen as a function of φ, it will turn out to work.

$$\begin{cases} \dot r = \frac{d}{dt}\frac{1}{u(\varphi)} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\varphi}\overbrace{\frac{d\varphi}{dt}}^{\dot\varphi = \frac{l}{\mu r^2} = u^2\frac{l}{\mu}} = -\frac{l}{\mu}\frac{du}{d\varphi} \\ \ddot r = \frac{d}{dt}\left(-\frac{l}{\mu}\frac{du}{d\varphi}\right) = \frac{d}{d\varphi}\left(-\frac{l}{\mu}\frac{du}{d\varphi}\right)\underbrace{\frac{d\varphi}{dt}}_{u^2\frac{l}{\mu}} = -\frac{l^2}{\mu^2}u^2\frac{d^2u}{d\varphi^2} \end{cases}$$

$$r\left(\frac{l}{r^2\mu}\right)^2 - \frac{GM}{r^2} = \ddot{r} \quad (r)$$

Insert in (r)

$$u^3 \frac{L^2}{\mu^2} - GMu^2 = -\underbrace{\frac{l^2}{\mu^2}}\, u^2 \frac{d^2u}{d\varphi^2}$$

Divide out $\frac{l^2}{\mu^2} u^2$

$$\frac{d^2u}{d\varphi^2} + \underbrace{u - \frac{GM\mu^2}{l^2}}_{w} = 0$$

with $w = u - \frac{GM\mu^2}{l^2}$ we have (get rid of const term)

$$\frac{d^2w}{d\varphi^2} + w = 0 \quad \text{(harmonic oscillator)}$$

$$\Rightarrow w = A\cos(\varphi - \underbrace{\delta}_{\text{some phase}})$$

Pick the pase $\delta = 0$

$$\Rightarrow u = A\cos\varphi + \frac{GM\mu^2}{l^2} \Rightarrow$$

$$r = \frac{1}{u} = \frac{1}{A\cos\varphi + \frac{GM\mu^2}{l^2}} = \frac{l^2}{GM\mu^2}\,\frac{1}{1 + \frac{l^2A}{GM\mu^2}\cos\varphi}$$

$\frac{l^2A}{GM\mu^2} \equiv \varepsilon$ the <u>eccentricity</u>

For maximal $\cos\varphi$, $\varphi = 0$ we get <u>perihelion</u>

$$r_{min} = \frac{l^2}{GM\mu^2}\,\frac{1}{1+\varepsilon} \quad (r_{min})$$

and similarly for <u>aphelion</u>

$$r_{max} = \frac{l^2}{GM\mu^2}\,\frac{1}{1-\varepsilon}$$

About one page of calculations in Cartesian coordinates (x,y) gives
(see last page)

- $\varepsilon = 0$ $x^2 + y^2 = r_{min}^2$ circle
- $0 \leq \varepsilon < 1$ same sign for x^2 and y^2 ellipse
- $\varepsilon = 1$ no x^2-term parabola
- $\varepsilon > 1$ different signs for x^2 and y^2 hyperbola

The solutions are conic sections.
kägelsnitt

In particular we note:

Kepler's 1st law: The orbit of every planet is an ellipse with the sun at one of the two focal points.

Consider $\varepsilon=0$ (circle) from r_{min} we have $l = \mu\sqrt{GMr_{min}}$, $r_{min} = r$

using $l = \mu r^2 \dot{\varphi}$ and $r = r_{min}$ we get

$$\omega^2 = \dot{\varphi}^2 = \frac{l^2}{\mu^2 r^4} = \frac{1}{\cancel{\mu^2} r^{\cancel{4}\,3}} \cancel{\mu^2} G M \cancel{r}$$

$$\omega^2 = \left(\frac{2\pi}{\tau}\right)^2$$ orbital period

$$\tau = \frac{2\pi}{\omega}$$

For planets $M \approx M_{sun}$

$$\Rightarrow \tau^2 = \left(\frac{2\pi}{\omega}\right)^2 = \frac{4\pi^2 r^3}{G M_{sun}}$$ i.e. $\tau^2 \propto r^3$ Kepler's 3rd law

We have seen that l is conserved, but we also know

1) L does not depend explicitly on time

2) The potential is not velocity dependent

3) The coordinate transformation from $\bar{r}$ to r, φ does not depend explicitly on time.

$\Rightarrow$ The Hamiltonian is constant and the energy is also conserved.

The orbits can be characterized by two constants of motion l and E.

For E we have

$$E = \frac{\mu}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - \mu\frac{GM}{r}$$

$\dot{\varphi} = \frac{l}{\mu r^2} \rightarrow$
$$= \frac{\mu}{2}\dot{r}^2 + \frac{\mu}{2}\frac{l^2}{\mu^2 r^2} - \mu\frac{GM}{r}$$

$u = \frac{1}{r},\ \dot{r} = -\frac{l}{\mu}\frac{du}{d\varphi}$
$$= \frac{\mu}{2}\frac{l^2}{\mu^2}\left(\frac{du}{d\varphi}\right)^2 + \frac{1}{2}\frac{l^2}{\mu}u^2 - \mu GMu$$

$u = A\cos\varphi + \frac{GM\mu^2}{l^2}$
$$= \frac{l^2}{2\mu}A^2\sin^2\varphi + \frac{l^2}{2\mu}\left(A\cos\varphi + \frac{GM}{l^2}\mu^2\right)^2 - \mu GM\left(A\cos\varphi + \frac{GM\mu^2}{l^2}\right)$$

$\sin^2 + \cos^2 = 1 \rightarrow$
$$= \frac{l^2}{2\mu}A^2 + \frac{\cancel{l^2}}{\cancel{2}\cancel{\mu}}\cancel{2}A\cos\varphi\frac{GM\mu^{\cancel{2}}}{\cancel{l^2}} + \frac{1}{2}\frac{G^2M^2\mu^3}{l^2} - \mu GMA\cos\varphi - \frac{G^2M^2\mu^3}{l^2}$$

$$= \frac{l^2}{2\mu}A^2 - \frac{1}{2}\frac{G^2M^2}{l^2}\mu^3$$

$$\Rightarrow\quad A^2 = \frac{G^2M^2}{l^4}\mu^4 + \frac{2E}{l^2}\mu$$

Giving for the amplitude

$$\Rightarrow\quad A = \underset{(A>0)}{\pm}\frac{GM\mu^2}{l^2}\sqrt{1 + 2\frac{l^2}{M^2G^2}\frac{E}{\mu^3}}$$

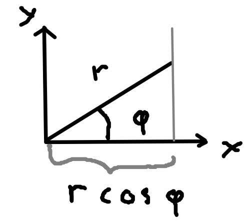
From $\varepsilon = \frac{l^2A}{GM\mu^2}$ we get $\varepsilon = \sqrt{1 + \frac{2\,l^2}{G^2M^2}\frac{E}{\mu^3}}$

$$\Rightarrow \varepsilon^2 = 1 + \frac{2 l^2}{G^2 M^2} \frac{E}{\mu^3}$$

> 0

- $0 \le \varepsilon < 1$ ellipse $E < 0$
- $\varepsilon = 1$ parabola $E = 0$
- $\varepsilon > 1$ hyperbola $E > 0$

Orbits:



Let $r = \frac{\overset{= \frac{l^2}{GM\mu^2}}{c}}{1+\varepsilon\cos\varphi} = \frac{c}{1+\varepsilon\frac{x}{r}} = \frac{cr}{r+\varepsilon x}$

$$\Leftrightarrow \quad r + \varepsilon x = c$$

$$\Rightarrow \quad r^2 = (c-\varepsilon x)^2 = c^2 - 2\varepsilon x c + \varepsilon^2 x^2$$

$$\Leftrightarrow \quad x^2(1-\varepsilon^2) + y^2 + 2\varepsilon x c = c^2$$

Note

- $\varepsilon = 1$ parabola $y^2 + 2\varepsilon x c = c^2$
- $\varepsilon > 1$ hyperbola $x^2\underbrace{(1-\varepsilon^2)}_{<0} + y^2 + 2\varepsilon x c = c^2$
- $0 \le \varepsilon < 1$ ellipse $x^2\underbrace{(1-\varepsilon^2)}_{>0} + y^2 + 2\varepsilon x c = c^2$

Study the ellipses:

$$\overset{0\le\varepsilon<1}{\Rightarrow} \quad \left(x\sqrt{1-\varepsilon^2} + \frac{\varepsilon c}{\sqrt{1-\varepsilon^2}}\right)^2 - \frac{\varepsilon^2 c^2}{1-\varepsilon^2} + y^2 = c^2$$

$$\Leftrightarrow \quad \left(x\sqrt{1-\varepsilon^2} + \frac{\varepsilon c}{\sqrt{1-\varepsilon^2}}\right)^2 + y^2 = c^2\left(1 + \frac{\varepsilon^2}{1-\varepsilon^2}\right) = c^2\left(\frac{1-\varepsilon^2+\varepsilon^2}{1-\varepsilon^2}\right) = c^2\frac{1}{1-\varepsilon^2}$$

$$\Rightarrow \quad \frac{\left(x + \frac{\varepsilon c}{1-\varepsilon^2}\right)^2(1-\varepsilon^2)}{\frac{c^2}{1-\varepsilon^2}} + \frac{y^2}{\frac{c^2}{1-\varepsilon^2}} = 1$$

$$\Leftrightarrow \quad \frac{\left(x + \overbrace{\frac{\varepsilon c}{1-\varepsilon^2}}^{\equiv d}\right)^2}{\Big(\underbrace{\frac{c}{1-\varepsilon^2}}_{\equiv a}\Big)^2} + \frac{y^2}{\Big(\underbrace{\frac{c}{\sqrt{1-\varepsilon^2}}}_{\equiv b}\Big)^2} = 1$$