Taylor: 521-533

The Hamilton formalism

So far we have encountered the Newtonian formulation of mechanics and the Lagrangian. Now we will study yet another formalism, the Hamilton formalism, where

$$H(\bar{q},\bar{p}) = \sum_{i=1}^{n} p_i \dot{q}_i(\bar{q},\bar{p}) - L$$

is the basic object. Note that we view H as depending on the (generalized coordinates $\bar{q}$ and) generalized momenta $\bar{p}$, rather than generalized velocities $\dot{q}_i$.

The 2N-dimensional space given by $\{\bar{q},\bar{p}\}$ we call the phase space.

For simplicity, we start with the case of one generalized coordinate

$$L = L(\bar{q},\dot{\bar{q}},t) = T(\bar{q},\dot{\bar{q}},t) - V(\bar{q},t)$$

Red: Several generalized coordinates, skip at first reading

In natural coordinates our Lagrangian has the form

Still no time dependence for natural coordinates

$$L = \frac{1}{2} A(q)\dot{q}^2 - V(q)$$

$$L = \frac{1}{2}\sum_{i,j} A_{ij}(\bar{q})\dot{q}_i\dot{q}_j - V(\bar{q},t) = T - V$$

From the definition of p we have

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \sum_j A_{ij}(\bar{q})\dot{q}_j$$

meaning that the Hamiltonian is

$$H = \sum_i p_i\dot{q}_i - L = \underbrace{\sum_{i,j} A_{ij}(\bar{q})\dot{q}_i\dot{q}_j}_{2T} - \Big(\underbrace{\sum_{i,j}\frac{1}{2}A_{ij}(q)\dot{q}_i\dot{q}_j}_{T} - V(\bar{q})\Big)$$

$$= 2T - (T - V) = T + V = E$$

i.e. for natural systems the Hamiltonian is just the energy.

To view H as depending on $\bar{p}$ rather than $\dot{\bar{q}}$ we solve for $\dot{\bar{q}}$,

$$\dot{q} = \frac{p}{A(q)} = \dot{q}(q, p)$$

$$H(\bar{q}, \bar{p}) = \sum_i p_i \dot{q}_i(\bar{q}, \bar{p}, t) - L\left(\bar{q}, \dot{\bar{q}}(\bar{q}, \bar{p}, t)\right)$$

From this Hamiltonian we will derive a new set of equations

describing the motion, Hamiltons equations.

First we differentiate H w.r.t. q:

each $\dot{q}_j$ may depend on q_i

explicit q_i dependence

implicit dependence since each $\dot{q}_j$ may depend on q_i

$$\frac{\partial H}{\partial q_i} = \cancel{\sum_j p_j \frac{\partial \dot{q}_j}{\partial q_i}} - \left[\frac{\partial L}{\partial q_i} + \cancel{\sum_j \underbrace{\frac{\partial L}{\partial \dot{q}_j}}_{\equiv p_j} \frac{\partial \dot{q}_j}{\partial q_i}}\right]$$

$$= -\frac{\partial L}{\partial q_i} = -\frac{d}{dt}\underbrace{\left(\frac{\partial L}{\partial \dot{q}_i}\right)}_{\equiv p_i} = -\dot{p}_i$$

Next, differentiate w.r.t. p instead

$$\frac{\partial H}{\partial p_i} = \dot{q}_i + \cancel{\sum_j p_j \frac{\partial \dot{q}_j}{\partial p_i}} - \cancel{\sum_j \underbrace{\frac{\partial L}{\partial \dot{q}_j}}_{\equiv p_j} \frac{\partial \dot{q}_j}{\partial p_i}} = \dot{q}_i$$

In total we have Hamilton's equations

$$\boxed{\frac{\partial H}{\partial q_i} = -\dot{p}_i} \qquad \boxed{\frac{\partial H_i}{\partial p} = \dot{q}_i}$$

Note: In the Lagrangian approach we had ~~one~~ n 2nd order equation.

In the Hamiltonian approach we have ~~two~~ 2n 1st order equations.

Hamilton's equations in several variables

Let's assume that we have as many coordinates as the system has d.o.f.

we have what in Taylor is known as holonomic constraints

forces can be derived from the potential energy.

(We may let the coordinate transformation from Cartesian to generalized coordinates depend on time.)

Consider the Lagrangian

$$L = L(\bar{q}, \dot{\bar{q}}, t) = T - V$$

where $\begin{cases} \bar{q} = (q_1, \cdots q_n) \\ \dot{\bar{q}} = (\dot{q}_1, \cdots \dot{q}_n) \end{cases}$

We have the Euler-Lagrange equations

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \qquad i = 1 \cdots n$$

and the Hamiltonian is

$$H \equiv \sum_{i=1}^{n} p_i \dot{q}_i - L$$

where $$p_i = \frac{\partial L(\bar{q}, \dot{\bar{q}}, t)}{\partial \dot{q}_i} \qquad i = 1 \cdots n$$

Now we want to express our Hamiltonian in the 2n variables $\bar{q}, \bar{p}$, thus

we rewrite: $$\dot{q}_i = \dot{q}_i(q_1 \cdots q_n, p_1 \cdots p_n, t) \qquad i = 1 \cdots n$$

or: $$\dot{\bar{q}} = \dot{\bar{q}}(\bar{q}, \bar{p}, t)$$

giving us the Hamiltonian

$$H = H(\bar{q}, \bar{p}, t) = \sum_{i=1}^{n} p_i \dot{q}_i(\bar{q}, \bar{p}, t) - L\left(\bar{q}, \dot{\bar{q}}(\bar{q}, \bar{p}, t), t\right)$$

Similar to the case of one variable one can derive (see red above)

$$\boxed{\dot{q}_i = \frac{\partial H}{\partial p_i}} \qquad \boxed{\dot{p}_i = -\frac{\partial H}{\partial q_i}}$$

Hamilton's equations

Consider the time derivative of H.

Naively H can vary with time because

- H changes as the coordinates change with time
- H changes if it depends explicitly on time

s.t.

$$\underbrace{\frac{dH}{dt}}_{\text{total derivative}} = \sum_{i=1}^{n}\left(\frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i\right) + \underbrace{\frac{\partial H}{\partial t}}_{\text{partial derivative}}$$

However, due to Hamilton's equations the first term vanishes for each i

$$\left(\frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i\right) = \cancel{\frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i}} + \cancel{\frac{\partial H}{\partial p_i}\left(-\frac{\partial H}{\partial q_i}\right)} = 0 \quad \forall i$$

i.e. H only changes with time if it depends explicitly on time.

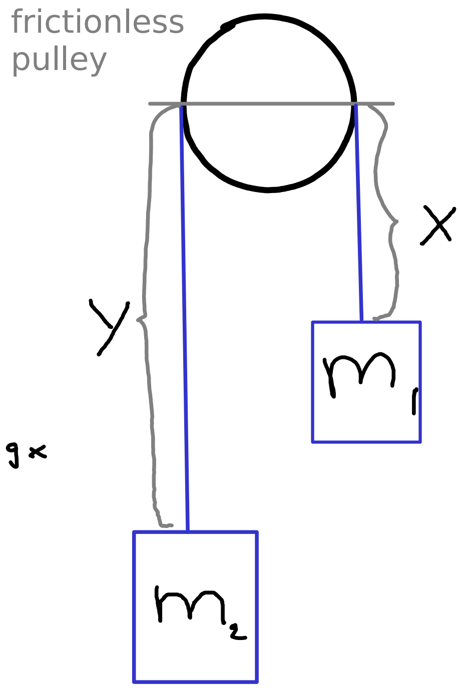
$$\frac{dH}{dt} = \frac{\partial H}{\partial t}$$

In particular, if H does not depend explicitly on time H is conserved.

Recall that for natural coordinates H=E, so energy is conserved if
(the relation between the generalized coordinates and Cartesian coordinates does not involve time)

H does not depend explicitly on time.

Example: The Atwood machine



For the Lagrangian we have

$$L = T - V = \frac{1}{2}(m_1+m_2)\dot{x}^2 - -(m_1-m_2)gx$$

$$H \equiv p\dot{x} - L =$$

$$= \left(\frac{\partial L}{\partial \dot{x}}\right)\dot{x} - L$$

$$= (m_1+m_2)\dot{x}^2 - \frac{1}{2}(m_1+m_2)\dot{x}^2 - (m_1-m_2)gx$$

$$= \frac{1}{2}(m_1+m_2)\dot{x}^2 - (m_1-m_2)gx$$

Indeed H=E, as it should be for natural systems.

Alternatively, knowing that we have natural generalized coordinates,

we could have written down

$$H = E = T + V = \frac{1}{2}(m_1+m_2)\dot{x}^2 - (m_1-m_2)gx$$

To use Hamilton's equations, we first calculate the generalized momentum

$$p = \frac{\partial L}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}} = (m_1+m_2)\dot{x}$$

To write down H in terms of q and p, we solve for $\dot{x} = \frac{p}{m_1+m_2}$

$$\Rightarrow H(x,p) = \frac{1}{2}(m_1+m_2)\left(\frac{p}{m_1+m_2}\right)^2 - (m_1-m_2)gx$$

$$= \frac{1}{2}\frac{p^2}{m_1+m_2} - (m_1-m_2)gx$$

Giving the Hamilton equations

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m_1+m_2}, \qquad \dot{p} = -\frac{\partial H}{\partial x} = - -(m_1-m_2)g$$

$$\Downarrow$$

$$\underline{\ddot{x}} = \frac{\dot{p}}{m_1+m_2} = \underline{\frac{(m_1-m_2)g}{m_1+m_2}}$$ as before

Hamilton's equations for a particle in a central force field

Find Hamilton's equations for a particle of mass m moving in

a central force field V(r).

The kinetic energy is

$$T = \frac{1}{2} m (\dot r^2 + r^2 \dot\varphi^2)$$

giving

$$p_r = \frac{\partial L}{\partial \dot r} = m\dot r \qquad p_\varphi = \frac{\partial L}{\partial \dot\varphi} = m r^2 \dot\varphi$$

m*(radial velocity) (angular momentum)

To express H in terms of $\bar q, \bar p$ we solve for $\dot r$, $\dot \varphi$

$$\dot r = \frac{p_r}{m} \qquad \dot\varphi = \frac{p_\varphi}{mr^2}$$

$$\Rightarrow H = T + V = \frac{1}{2} m \left[\left(\frac{p_r}{m}\right)^2 + r^2 \left(\frac{p_\varphi}{mr^2}\right)^2 \right] + V(r)$$

$$= \frac{1}{2m}\left[p_r^2 + \frac{p_\varphi^2}{r^2} \right] + V(r)$$

From H we get 2*2 1st order Hamilton equations:

$$\dot\varphi = \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{mr^2} \qquad \dot\varphi$$

(reproduces def. of p_φ)

$$\dot p_\varphi = -\frac{\partial H}{\partial \varphi} = 0 \qquad \dot p_\varphi$$

(angular momentum p_φ is conserved)

$$\dot r = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \qquad \dot r$$

(reproduces def of radial momentum)

$$\dot p_r = -\frac{\partial H}{\partial r} = -\frac{(-\cancel{2})}{\cancel{2}}\frac{p_\varphi^2}{mr^3} - \frac{\partial V}{\partial r} \qquad \dot p_r$$

(most interesting equation)

If we substitute the derivative of $\dot r$ into $\dot p_r$ we get

$$\dot p_r \overset{\dot r}{=} \ddot r m \Rightarrow m\ddot r \overset{\dot p_r}{=} \frac{p_\varphi^2}{mr^3} - \frac{\partial V}{\partial r}$$

centrifugal force actual force

In general the procedure for setting up Hamilton's equations are:

1) Choose generalized coordinates

2) Write down T and V in terms of $\bar{q}, \dot{\bar{q}}$

V independent of $\dot{q}_i$

3) Find the generalized momenta $p_i = \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i}$

4) Solve for $\dot{q}_i$ in terms of $\bar{q}, \bar{p}$

5) Write down the Hamiltonian in terms of $\bar{q}, \bar{p}$
If the coordinates are natural H=T+V.

6) Write down and solve Hamilton's equations

Cyclic (ignorable) coordinates

We have seen for the Lagrangian that if L is independent of q_i

(i.e. q_i is a cyclic or ignorable coordinate)

the corresponding conjugated momentum $p_i = \frac{\partial L}{\partial \dot{q}_i}$ is conserved.

In the same way it follows from Hamiltons equations $\dot{p}_i = -\frac{\partial H}{\partial q_i}$

that if H is independent of q_i the corresponding p_i is conserved.

This is actually the same result since

$$H(\bar{q},\bar{p},t) = \sum_{i=1}^{n} p_i \dot{q}_i(\bar{q},\bar{p},t) - L(\bar{q}, \dot{\bar{q}}(\bar{q},\bar{p},t),t)$$

$$\frac{\partial H}{\partial q_i} = \overbrace{\sum_j \underbrace{\frac{\partial}{\partial \dot{q}_j}(p_j\dot{q}_j)}_{p_j} \frac{\partial \dot{q}_j}{\partial q_i}}^{\text{implicit via each } \dot{q}_j} - \overbrace{\frac{\partial L}{\partial q_i}}^{\text{explicit}} - \overbrace{\sum_j \underbrace{\frac{\partial L}{\partial \dot{q}_j}}_{p_j} \frac{\partial \dot{q}_j}{\partial q_i}}^{\text{implicit via each } \dot{q}_j} = -\frac{\partial L}{\partial q_i}$$

meaning that H is independent of q_i iff L is independent of q_i

For cyclic coordinates the Hamiltonian formalism has an advantage over the Lagrangian:

Let $H = H(q_1, q_2, p_1, p_2)$ where q_2 is a cyclic coordinate, s.t.:

$$H = H(q_1, p_1, p_2)$$

but since q_2 is cyclic $\dot{p}_2 = -\frac{\partial H}{\partial q_2} = 0 \Rightarrow p_2 = \text{const}$

$$\Rightarrow H = H(q_1, p_1, \text{const}) = H(q_1, p_1)$$

so the system has reduced to a one-dimensional system.

Note that this is not true in the Lagrangian formalism where

$$L = L(q_1, \cancel{q_2}, \dot{q}_1, \dot{q}_2) = L(q_1, \dot{q}_1, \dot{q}_2)$$

since $\dot{q}_2$ is not necessarily constant.