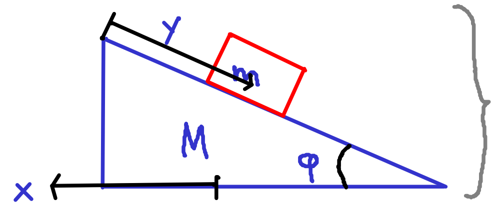
Ex: Hamilton formalism for block on wedge

We have seen before that



(1)

(2)
$$T = \frac{M}{2}\dot{x}^2 + \frac{m}{2}\left(\dot{y}^2 + \dot{x}^2 - 2\dot{x}\dot{y}\cos\varphi\right)$$

$$V = -mgy\sin\varphi$$

Since our coordinates do not depend explicitly on time we have

$$H = T + V = \frac{M}{2}\dot{x}^2 + \frac{m}{2}\left(\dot{y}^2 + \dot{x}^2 - 2\dot{x}\dot{y}\cos\varphi\right) - mgy\sin\varphi$$

Next we need the generalized momenta

$$p_x = \frac{\partial L}{\partial \dot{x}} = (M+m)\dot{x} - m\dot{y}\cos\varphi \qquad (x)$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} - m\dot{x}\cos\varphi \qquad (y)$$

(3)

We also know $\dot{p}_x = -\frac{\partial H}{\partial x} = 0 \Rightarrow p_x$ is a constant of motion

If the block and wedge start at rest $p_x = 0$ $\quad \dot{x} \overset{(x)}{=} \frac{m\cos\varphi}{M+m}\dot{y}$

Now we get from (y)

$$\Rightarrow \quad p_y = m\dot{y} - m\frac{m\cos\varphi}{M+m}\dot{y}\cos\varphi = \frac{m(M+m) - m^2\cos^2\varphi}{M+m}\dot{y}$$

$$= \frac{mM + m^2\sin^2\varphi}{M+m}\dot{y}$$

(4)

$$\dot{y} = \frac{(M+m)}{m(M+m\sin^2\varphi)}p_y$$

$$\dot{x} = \frac{\cancel{m}\cos\varphi}{(\cancel{M+m})}\frac{(\cancel{M+m})p_y}{\cancel{m}(M+m\sin^2\varphi)} = \frac{\cos\varphi\, p_y}{M+m\sin^2\varphi}$$

Write down H in terms of $\bar{q}, \bar{p}$

$$H = T + V = \frac{M}{2}\dot{x}^2 + \frac{m}{2}\left(\dot{y}^2 + \dot{x}^2 - 2\dot{x}\dot{y}\cos\varphi\right) - mgy\sin\varphi$$

$$= \frac{p_y^2}{(M + m\sin^2\varphi)^2}\left[\frac{M}{2}\cos^2\varphi + \frac{\cancel{m}}{2}\frac{(M+m)^2}{m^{\cancel{2}}} + \frac{m}{2}\cos^2\varphi - \cancel{m}\cos^2\varphi\,\frac{M+m}{\cancel{m}}\right] - mgy\sin\varphi$$

$$= \frac{p_y^2}{(M + m\sin^2\varphi)^{\cancel{2}}}\underbrace{\left[-\frac{M+m}{2}\cos^2\varphi + \frac{(M+m)^2}{2m}\right]} - mgy\sin\varphi$$

$$\frac{(M+m)}{2}\left[-\cos^2\varphi + \frac{M+m}{m}\right]$$

$$\frac{(M+m)}{2m}\Big[\underbrace{-m\cos^2\varphi + M + m}_{M + m\sin^2\varphi}\Big]$$

$$= \frac{1}{2m}\frac{p_y^2 (M+m)}{M + m\sin^2\varphi} - mgy\sin\varphi$$

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Note that $H = H(y, p_y)$ whereas $L = L(\cancel{x}, \dot{x}, y, \dot{y})$

The Hamilton equations are:

$$\boxed{\dot{q}_i = \frac{\partial H}{\partial p_i}} \qquad \boxed{\dot{p}_i = -\frac{\partial H}{\partial q_i}}$$

$$\dot{p}_y = -\frac{\partial H}{\partial y} = mg\sin\varphi$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m}\,\frac{(M+m)}{(M + m\sin^2\varphi)}$$

Differentiate

$$\Rightarrow \ddot{y} = \frac{g\cancel{m}\sin\varphi}{\cancel{m}}\,\frac{(M+m)}{(M + m\sin^2\varphi)} \quad \Rightarrow \ddot{x} = \frac{g\,m\sin\varphi\cos\varphi}{(M + m\sin^2\varphi)}$$

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Poisson brackets (advanced)

We have seen the Hamilton equations

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}$$

These equations are almost symmetric, but there is an annoying minus sign....

It turns out that this can be written in an even more symmetric form

if we introduce Poisson brackets. In general for two functions f(q,p), g(q,p)

the poisson bracket is:

$$\{f(q,p), g(q,p)\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial g}{\partial q}\frac{\partial f}{\partial p}$$

Then we have

$$\boxed{\{q, H\}} = \underbrace{\frac{\partial q}{\partial q}}_{1}\frac{\partial H}{\partial p} - \frac{\partial H}{\partial q}\underbrace{\frac{\partial q}{\partial p}}_{0} = \frac{\partial H}{\partial p} = \boxed{\dot{q}}$$

$$\boxed{\{p, H\}} = \underbrace{\frac{\partial p}{\partial q}}_{0}\frac{\partial H}{\partial p} - \frac{\partial H}{\partial q}\underbrace{\frac{\partial p}{\partial p}}_{1} = -\frac{\partial H}{\partial q} = \boxed{\dot{p}}$$

(We don't need to remember where the minus is.)

i.e., Hamilton's equations can alternatively be written in completely

symmetric form.

Note that the time derivative of a general function can be written in terms of H

$$\frac{d}{dt} f(q,p,t) = \underbrace{\frac{\partial f}{\partial q}\underbrace{\underbrace{\frac{\partial q}{\partial t}}_{\dot{q}}}_{\frac{\partial H}{\partial p}} + \frac{\partial f}{\partial p}\underbrace{\underbrace{\frac{\partial p}{\partial t}}_{\dot{p}}}_{-\frac{\partial H}{\partial q}}}_{\{f, H\}} + \frac{\partial f}{\partial t}$$

$$= \{f, H\} + \frac{\partial f}{\partial t}$$

So if a function, for example momentum, angular momentum or energy

does not depend explicitly on time and has a vanishing Poisson bracket

with H, it is conserved.

Ex: Momentum in a constant potential

$$H = E = T + V = \frac{1}{2} m \dot{x}^2 + \text{const} = \frac{1}{2} m \frac{p_x^2}{m^2} + \text{const}$$

$$\frac{dp_x}{dt} = \{p_x, H\} = \underbrace{\frac{\partial p_x}{\partial x}}_{0} \frac{\partial H}{\partial p_x} - \underbrace{\frac{\partial H}{\partial x}}_{0} \frac{\partial p_x}{\partial p_x} = 0 \Rightarrow p_x \text{ conserved}$$

Ex: The Hamiltonian itself

$$\frac{dH}{dt} = \underbrace{\{H, H\}}_{0} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$$

For a system with natural coordinates we have $E = H$

$\Rightarrow$ Energy is conserved if H does not depend explicitly on time.

Bending in a gravitational field

Taylor: 572-576
(Taylor integrates the force to find the scattering angle)

If a celestial body has energy >0 it follows a hyperbola.

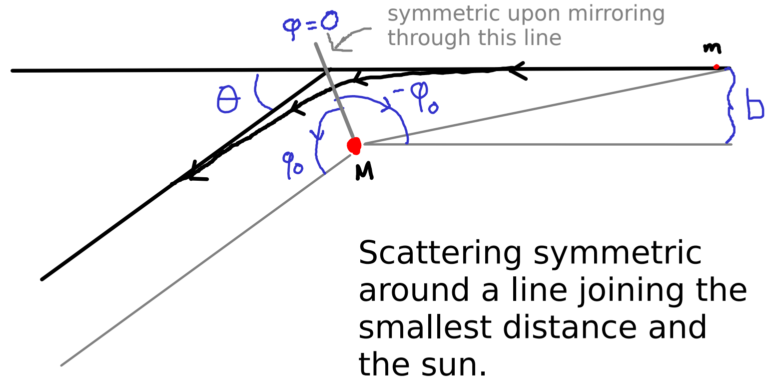
Let's study the angle by which it is deflected when passing by the sun.

We have seen

L5:5
Kep 5

$$r = r_{min} \frac{(1+\varepsilon)}{1+\varepsilon\cos\varphi}$$

r smallest for $\varphi = 0$



Let the $\varphi = -\varphi_0$ from the beginning

when $r \to \infty$ far away.

Scattering symmetric around a line joining the smallest distance and the sun.

After the passage we have $\varphi = +\varphi_0$

$r \to \infty$

$$\theta = 2\varphi_0 - \pi \Rightarrow \varphi_0 = \frac{\theta + \pi}{2}$$

corresponds to moving straight on

When $r \to \infty$ (afterwards) the denominator must $\to 0$, that is

$$\Rightarrow 0 = 1 + \varepsilon\cos(\varphi_0) = 1 + \varepsilon \underbrace{\cos\left(\frac{\theta+\pi}{2}\right)}_{-\sin\frac{\theta}{2}}$$

$$\Rightarrow \quad \sin\frac{\theta}{2} = \frac{1}{\varepsilon}$$

But we know the eccentricity

L5:7
Kep 7

$$\varepsilon = \sqrt{1 + \frac{2L^2}{G^2M^2}\frac{E}{\mu^3}}$$

angular momentum

Recall: $\mu = \frac{m_1 m_2}{m_1 + m_2} \overset{m_1 >> m_2}{\approx} m_2$

Using that the energy and angular momentum are conserved

and assuming initial speed v_i s.t.

$$E = \frac{1}{2} v_i^2 \mu \qquad \text{and} \qquad L = v_i b \mu$$

impact parameter

we can replace the eccentricity:

$$\Rightarrow \sin\frac{\theta}{2} = \frac{1}{\varepsilon} = \Bigg(1 + \overbrace{\frac{\cancel{2} v_i^2 b^2 \cancel{\mu}^2}{G^2 M^2}}^{L^2}\,\overbrace{\frac{1}{\cancel{2}} v_i^2 \cancel{\mu}}^{E} \frac{}{\cancel{\mu}^3}\Bigg)^{-1/2} = \left(1 + \frac{v_i^4 b^2}{G^2 M^2}\right)^{-1/2}$$

We thus know the scattering angle for given initial velocity and impact parameter.

Alternatively we can relate θ to the minimal distance using energy conservation:

$$\frac{1}{2} v_{max}^2 \cancel{\mu} - \frac{GM}{r_{min}} \cancel{\mu} = \frac{1}{2} v_i^2 \cancel{\mu}$$

energy closest=
energy furthest away

Giving $L^2 = (v_{max} r_{min})^2 \mu^2 = \Bigg[\overbrace{\frac{2GM}{r_{min}} + v_i^2}^{v_{max}^2}\Bigg] r_{min}^2 \mu^2$

so ε:

L5:7
Kep 7

$$\varepsilon = \sqrt{1 + \frac{2L^2}{G^2M^2}\frac{E}{\mu^3}} \overset{\text{insert } E \text{ and } L^2}{=} \sqrt{\underbrace{1 + \frac{\cancel{2}}{G^2M^2\cancel{\mu}^3} \overbrace{\left[\frac{2GM}{r_{min}} + v_i^2\right] r_{min}^2 \cancel{\mu}^2}^{L^2}\, \overbrace{\frac{1}{\cancel{2}} v_i^2 \cancel{\mu}}^{E}}_{\frac{2 r_{min} v_i^2}{GM} + \frac{(v_i^2)^2}{G^2M^2} r_{min}^2}}$$

$$\sqrt{(\;)^2} \to\; = 1 + \frac{v_i^2 r_{min}}{GM}$$

$$\Rightarrow \sin\frac{\theta}{2} = \frac{1}{\varepsilon} = \frac{1}{1 + \frac{v_i^2 r_{min}}{GM}}$$

If this was valid for light as well it would be possible to calculate the bending of light around the sun. A correct description according to general relativity gives a factor 2 larger deviation. (Measured 1919)

Cross sections and Rutherford's scattering formula

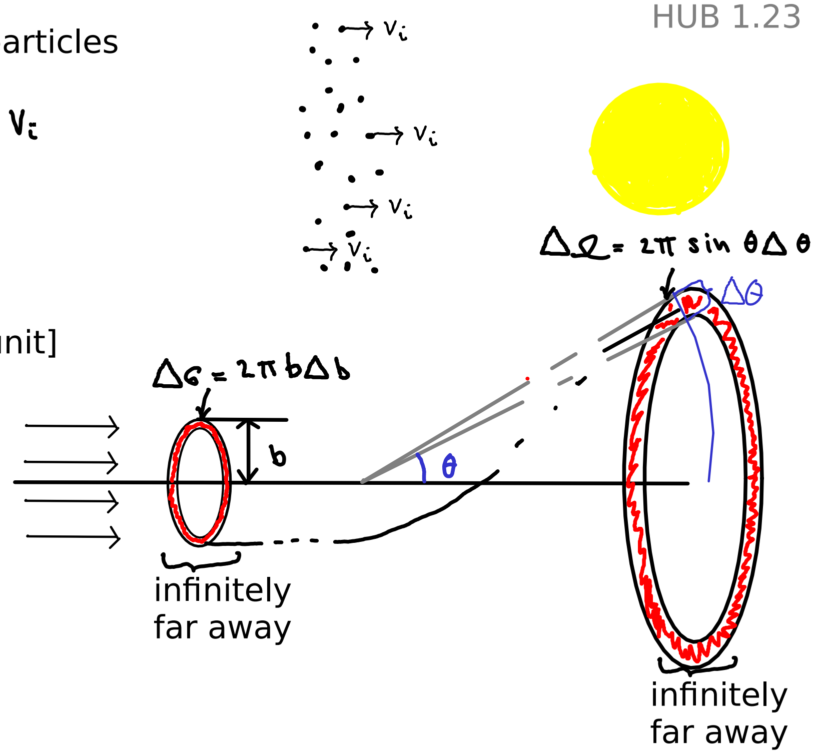
Consider a homogeneous flow of particles

approaching the sun with velocity v_i

We want to investigate

[the number of particles leaving

within solid angle $\Delta\Omega$ per time unit]



For $\Delta\Omega$ we have

$$\Delta\Omega = \underbrace{2\pi \sin\theta}_{\text{circumference}} \Delta\theta$$

The number of particles must be

proportional to the intensity, I ,

I =[the number of particles approacing the sun per unit area and unit time]

Assume that we have $N = \Delta\sigma I$ particles entering $\Delta\sigma$ per unit time.

These particles then leave in $\Delta\Omega$ giving

$$N = \Delta\sigma \cdot I = 2\pi b \Delta b \, I$$

$$= \frac{\Delta\sigma}{\Delta\Omega} \Delta\Omega \, I = \frac{\Delta\sigma}{\Delta\Omega} \overbrace{2\pi \sin\theta \Delta\theta}^{\Delta\Omega} \cdot I$$

From we get

$$\frac{\Delta\sigma}{\Delta\Omega} = \frac{b}{\sin\theta} \frac{\Delta b}{\Delta\theta} \Rightarrow \frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta} \left| \frac{db}{d\theta} \right|$$

σ is called cross section.

- $\frac{d\sigma}{d\Omega} > 0$ increasing Ω => increasing σ
- $\sin\theta > 0$ for $0 < \theta < \pi$
- but $\frac{db}{d\theta} < 0$

We have seen the relation

L8:6
Bend2

$$\sin\frac{\theta}{2} = \left(1 + \frac{v_i^4 b^2}{G^2 M^2}\right)^{-1/2}$$

between the scattering angle and the impact parameter.

$$\Rightarrow \quad 1 + \frac{v_i^4 b^2}{G^2 M^2} = \frac{1}{\sin^2 \frac{\theta}{2}}$$

$$\Rightarrow \quad b^2 = \frac{G^2 M^2}{v_i^4} \left(\frac{1}{\sin^2 \frac{\theta}{2}} - 1 \right)$$

We want $b \frac{db}{d\theta}$ for $\frac{\Delta\sigma}{\Delta\Omega}$

$$b \frac{db}{d\theta} = \frac{1}{2} \frac{d}{d\theta} (b^2) = \frac{1}{2} \frac{G^2 M^2}{v_i^4} (-\cancel{2}) \frac{1}{\cancel{2}} \frac{\cos(\frac{\theta}{2})}{\sin^3 \frac{\theta}{2}}$$

$$\frac{d\sigma}{d\Omega} = \frac{b}{\underbrace{\sin\theta}_{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}} \left| \frac{db}{d\theta} \right| = \frac{1}{2 \sin \frac{\theta}{2} \cancel{\cos \frac{\theta}{2}}} \frac{1}{2} \frac{G^2 M^2}{v_i^4} \frac{\cancel{\cos(\frac{\theta}{2})}}{\sin^3 \frac{\theta}{2}}$$

$$= \frac{G^2 M^2}{4 v_i^4} \frac{1}{\sin^4 \frac{\theta}{2}}$$

$$= \frac{G^2 M^2 m^2}{16 (\frac{1}{2} m v_i^2)^2} \frac{1}{\sin^4 \frac{\theta}{2}}$$

$$= \left(\frac{G M m}{4E} \right)^2 \frac{1}{\sin^4 \frac{\theta}{2}}$$

Note that we essentially only used the $\frac{1}{r}$ -potential

$\Rightarrow$ For scattering in an electrostatic potential $\quad GMm \to \frac{Qq}{4\pi\varepsilon_0}$

$$\Rightarrow \quad \frac{d\sigma}{d\Omega} = \left(\frac{Qq}{16 \pi \varepsilon_0 E} \right)^2 \frac{1}{\sin^4 \frac{\theta}{2}}$$

Rutherford's scattering formula.