Analytical mechanics, repetition

The integral of a (continuously differentiable) function $\int_A^B f(y(x), y'(x))\,dx$

is minimized (maximized) by functions fulfilling

$$\frac{\partial f}{\partial y} = \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$$

Euler-Lagrange equation

where y and y' are considered to be independent.

If f depends on many variables $f(y_1 \cdots y_n, y_1' \cdots y_n')$ and their derivatives

we get one equation for each variable i,

$$\frac{\partial f}{\partial y_i} = \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right)$$

i.e., we have n Euler-Lagrange equations.

If we also let y_i depend on m integration variables $x_1 \cdots x_m$ and extremize

$$\int dx_1 \cdots dx_m f\left(y_1(x_1 \cdots x_m), \cdots, y_n(x_1 \cdots x_m), \frac{\partial y_1}{\partial x_1}(x_1 \cdots x_m), \cdots, \frac{\partial y_1}{\partial x_m}(x_1 \cdots x_m), \cdots, \frac{\partial y_n}{\partial x_1}(x_1 \cdots x_m), \cdots, \frac{\partial y_n}{\partial x_m}(x_1 \cdots x_m)\right)$$

we get n equations

$$\frac{\partial f}{\partial y_i} = \sum_{j=1}^{m} \frac{\partial^{tot}}{\partial x_j}\left(\frac{\partial f}{\partial\left(\frac{\partial y_i}{\partial x_j}\right)}\right)$$

In nature, the action, the time integral of the Lagrangian

$$S = \int \underbrace{(E_k - E_p)}_{L} dt$$

S: the action; L: the Lagrangian

is mimimized for classical physics, i.e., we have

$$0 = \delta S = \delta \int_A^B L\,dt$$ for physical solutions.

In particular, for a point particle in an $\dot{x}$ -independent potential

$$L(x,\dot{x}) = \frac{m\dot{x}^2}{2} - V(x),$$

and we get back Newton's second law of motion from the (Euler-) Lagrange equation

$$\boxed{\frac{\partial L}{\partial x} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)}$$

It is often useful to use other d.o.f. than $\vec{x}$ to describe a system,

for example angle and radius for a central motion.

Instead of Cartesian coordinates we may thus use generalized coordinates

to describe the relevant d.o.f.. (Think change of coordinates.)

The Lagrangian, L, may then be expressed as

$$L = L(q_1 \cdots q_n, \dot{q}_1 \cdots \dot{q}_n)$$

where q_i $i = 1 \ldots n$ are called generalized coordinates

Similarly $\dot{q}_i = \frac{dq_i}{dt}$ generalized velocities

$p_i = \frac{\partial L}{\partial \dot{q}_i}$ generalized momenta

$Q_i = \frac{\partial L}{\partial q_i}$ generalized forces

In these coordinates the principle of least action, Hamilton's principle, gives

$$\boxed{\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)} \qquad i = 1 \ldots n$$ (Euler-) Lagrange's equations

From this we have

$$\frac{d}{dt}(p_i) = \frac{\partial L}{\partial q_i}$$

This means that if L does not depend explicitly on q_i the corresponding

conjugated momentum is conserved!

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A generalized coordinate that does not explicitly enter the Lagrangian is called a cyclic or ignorable coordinate and the corresponding conserved quantity is called a constant of motion.

That we have a constant of motion when L is independent of the corresponding conjugated coordinate is a special case of Noether's theorem which states:

If a system has a continuous symmetry, there is a corresponding conserved quantity.

Euler-Lagrange equations are often non-linear coupled differential equations which are not analytically solvable. However, many problems can be solved approximately by linearizing the system around equilibrium points. (Double pendulum, 3-body problem etc.)

Often one gets a system of equations of the form

$$\begin{cases} \ddot{q}_1 = w_{11} q_1 + w_{12} q_2 + k_1 \\ \ddot{q}_2 = w_{21} q_1 + w_{22} q_2 + k_2 \end{cases}$$

homogeneous

inhomogeneous

and analogously in more dimensions.

We often find the solutions as a particular solution to the inhomogeneous stationary case $\dot{q}_1, \dot{q}_2 = 0$ $(\ddot{q}_1 = \ddot{q}_2 = 0)$ plus solutions to the homogeneous case.

The homogeneous system can be solved by the ansatz

$$q_{1,2} = A_{1,2}\, e^{i\Omega t}$$

$$\Rightarrow \begin{cases} (i\Omega)^2 A_1 = W_{11} A_1 + W_{12} A_2 \\ (i\Omega)^2 A_2 = W_{21} A_1 + W_{22} A_2 \end{cases}$$

$$\Rightarrow \begin{bmatrix} W_{11} + \Omega^2 & W_{12} \\ W_{21} & W_{22} + \Omega^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = 0$$

As the RHS is 0, the equations on the LHS must be linearly dependent for non-trivial solutions and thus $\det[\;\;] = 0$ gives the possible (frequencies) Ω.

We have also encountered optimization under a constraint (the hanging chain).

If we want to extremize an integral $\int_{x_1}^{x_2} f(y, y')\, dx$

under the constraint $\int_{x_1}^{x_2} g(y, y')\, dx = \text{const}$

this can be done by extreming $\int_{x_1}^{x_2} [f(y, y') + \lambda\, g(y, y')]\, dx$

where λ is a Lagrange-multiplier. The value of λ and integration constants are choosen such that the constraint and boundary conditions are fulfilled.

The Hamiltonian $H = \sum_{i=1}^{n} \left(\frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right) - L$

is conserved if $\frac{\partial L}{\partial t} = 0$ i.e. if L doesn't depend explicitly on time.

The Hamiltonian is (most often) seen as a function of

- generalized momenta (p_i)
- generalized coordinates (q_i)

} Hamilton's formulation of mechanics and quantum mechanics

In the Hamilton formalism we solve Hamilton's equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad , \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

If we have natural generalized coordinates, i.e., if the coordinate transformation between Cartesian and generalized coordinates does not depend explicitly on time, $H = E_k + E_p = E$.