1|

a)

$$E_k = \frac{1}{2} m v^2 = \frac{1}{2} m (L\dot\varphi)^2$$

$$E_p = -mgL\cos\varphi \quad (+\text{ const})$$

└ put E_p to 0 at suspension point $\Rightarrow$ const = 0

b)

Natural coordinates

$$H \overset{\downarrow}{=} E_k + E_p = \frac{1}{2} m L^2 \dot\varphi^2 - mgL\cos\varphi$$

$$p_\varphi = \frac{\partial H}{\partial \dot\varphi} = mL^2\dot\varphi \quad \Rightarrow \quad \dot\varphi = \frac{p_\varphi}{mL^2}$$

$$\Rightarrow H = \frac{1}{2} m L^2 \frac{p_\varphi^2}{(mL^2)^2} - mgL\cos\varphi$$

$$= \frac{1}{2}\frac{p_\varphi^2}{mL^2} - mgL\cos\varphi$$

c) Equilibrium point at $\varphi = 0$

Hamilton's equations:

$$\dot p_\varphi = -\frac{\partial H}{\partial \varphi} = --mgL(-\sin\varphi) \qquad \dot\varphi = \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{mL^2}$$

$$\Rightarrow \ddot\varphi = \frac{\dot p_\varphi}{mL^2} = -\frac{mg\cancel{L}\sin\varphi}{mL^{\cancel{2}}} \approx -\frac{g}{L}\varphi$$

$$\Rightarrow \varphi = A\cos\left(\sqrt{\frac{g}{L}}\,t + \delta\right)$$

⌈Check: $\ddot\varphi = -\frac{g}{L}\overbrace{A\cos\left(\sqrt{\frac{g}{L}}\,t+\delta\right)}^{\varphi}$ ok

2)

a)

$$\underline{P}_1 = \left(\frac{E_1}{c}, \bar{P}_1\right)$$

$$\underline{P}_2 = \left(\frac{E_2}{c}, \bar{P}_2\right)$$

Let $\underline{P}_1$ move in z-direction and $\underline{P}_2$ in -z-direction.

$$|\bar{P}_1| = \sqrt{\left(\frac{E_1}{c}\right)^2 - m^2c^2}$$

We have $E_1 = E_2 = E (= 6.5 \text{ TeV})$

$$\underline{P}_1 = \left(\frac{E}{c}, 0, 0, \sqrt{\left(\frac{E}{c}\right)^2 - (mc)^2}\right)$$

$$\underline{P}_2 = \left(\frac{E}{c}, 0, 0, -\sqrt{\left(\frac{E}{c}\right)^2 - (mc)^2}\right)$$

b) $$M^2c^2 = (\underline{P}_1 + \underline{P}_2)^2 = \left(\frac{2E}{c}\right)^2 = 4\frac{E^2}{c^2}$$

$$\Rightarrow M = \frac{2E}{c^2} = \frac{13\text{TeV}}{c^2}$$

c) Let proton 2 be at rest

$$\underline{P}_2' = (mc, 0, 0, 0)$$

$$\underline{P}_1' = \left(\frac{E'}{c}, 0, 0, \frac{E'}{c}\right) \qquad |\bar{P}_1'| \approx \frac{E'}{c}$$

$$\Rightarrow \underbrace{(\underline{P}_1' + \underline{P}_2')^2}_{\parallel \atop M^2c^2} = \left(\frac{E'}{c} + mc\right)^2 - \left(\frac{E'}{c}\right)^2 \approx 2\frac{E'}{\cancel{c}}mc$$

$$\Rightarrow E' = \frac{M^2c^2}{2m} = \left(13\frac{\text{TeV}}{c^2}\right)^2 \cdot \frac{c^2}{2m\frac{\text{GeV}}{c^2}}$$

$$= \frac{13^2\text{ TeV}^2}{2\cdot 10^{-3}\text{ TeV}} \approx 85\,000 \text{ TeV}$$

d)

Want $E_k \approx E' \approx 85000 \text{ TeV}$

Bumblebee is nonrelativistic

$$\Rightarrow E_k = \frac{mv^2}{2} \overset{!}{=} 85000 \text{ TeV}$$

$$\Rightarrow \quad v = \sqrt{\frac{2 \cdot 85000 \text{ TeV}}{0.2 \text{ g}}}$$

$$= \sqrt{85 \cdot 10^7 \text{ GeV} \frac{1}{\text{g}}}$$

$$= \sqrt{85 \cdot 10^7 \underbrace{\left(\frac{\text{GeV}}{c^2}\right)}_{1.8 \cdot 10^{-27} \text{ kg}} \cdot c^2 \frac{1}{10^{-3} \text{ kg}}}$$

$$= \sqrt{85 \cdot 10^7 \; 1.8 \cdot 10^{-27} \text{ kg} \underbrace{(3 \cdot 10^8)^2}_{10^{17}} \frac{\text{m}^2}{\text{s}^2} \frac{\cdot 10^3}{\text{kg}}}$$

$$= \sqrt{85 \cdot 10^{7-27+17+3} \cdot 1.8} \; \frac{\text{m}}{\text{s}}$$

$$\underbrace{\underbrace{\underbrace{}_{10^{-20}}}_{10^{-3}}}_{10^{0}}$$

$$= \sqrt{153} \; \frac{\text{m}}{\text{s}} \sim 12 \frac{\text{m}}{\text{s}} \sim 10 \frac{\text{m}}{\text{s}}$$

3a) In the frame of the voyager, the distance is Lorentz contracted a factor $\sqrt{1-\frac{v^2}{c^2}}$

$$\Rightarrow d' = \sqrt{1-\frac{v^2}{c^2}}\, d_0 \quad , \quad d_0 = 2.5\cdot10^6 \text{ ly}$$

(The Andromeda is moving towards the traveler with speed v.)

$$d' = t'v \Rightarrow \sqrt{1-\frac{v^2}{c^2}}\, d_0 = t'v$$

$$\Rightarrow \left(1-\frac{v^2}{c^2}\right) d_0^2 = t'^2 v^2 \Leftrightarrow v^2\left(t'^2 + \frac{d_0^2}{c^2}\right) = d_0^2$$

$$\Rightarrow v = \sqrt{\frac{d_0^2 c^2}{t'^2 c^2 + d_0^2}} = c\sqrt{\frac{1}{\frac{t'^2 c^2}{d_0^2} + 1}}$$

$$\approx c\left(1 - \frac{1}{2}\frac{t'^2 c^2}{d_0^2}\right)$$

$$= c\left(1 - \frac{1}{2}\frac{50^2 \cancel{y}^2 \cdot \cancel{c}^2}{\underbrace{(2.5\cdot10^6)}_{25\cdot10^5}{}^2 \cancel{y}^2 \cancel{c}^2}\right)$$

$$= c\left(1 - \frac{1}{2}(2\cdot10^{-5})^2\right) = c\left(1 - 2\cdot10^{-10}\right)$$

b) $v \approx c \Rightarrow \dfrac{c \frac{m}{s}}{10 \frac{m}{s^2}} = \dfrac{3\cdot10^8}{10} s = 3\cdot10^7 s$ ($\sim$ 1 year)

c) Rapidity is additive for parallel boosts.

Each second the rapidity is increased by

$$\frac{\Delta v}{c} = \frac{10\ \frac{m}{s}}{3\cdot 10^{8}\ \frac{m}{s}} = \frac{1}{3}\cdot 10^{-7}$$

It thus takes $\frac{11.5}{\frac{1}{3}\cdot 10^{-7}} = \underbrace{3\cdot 10^{7}}_{\sim\text{year}}\cdot 11.5 \approx 11$ years

4)

a) EL θ: $\dfrac{\partial L}{\partial \theta} = \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{\theta}}$

$$-\delta\theta - \frac{1}{2}\varepsilon z = I\ddot{\theta}$$

EL z: $\dfrac{\partial L}{\partial z} = \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{z}}\right)$

$$-kz - \frac{1}{2}\varepsilon\theta = m\ddot{z}$$

b) Write equations in matrix form

$$\begin{pmatrix} -\delta & -\frac{1}{2}\varepsilon \\ -\frac{1}{2}\varepsilon & -k \end{pmatrix}\begin{pmatrix} \theta \\ z \end{pmatrix} = \begin{pmatrix} I\ddot{\theta} \\ m\ddot{z} \end{pmatrix}$$

and make the ansatz

$$\begin{pmatrix} \theta \\ z \end{pmatrix} = \begin{pmatrix} A_\theta \\ A_z \end{pmatrix} e^{i\omega t}$$

$$\Rightarrow \begin{pmatrix} -\delta & -\frac{1}{2}\varepsilon \\ -\frac{1}{2}\varepsilon & -k \end{pmatrix}\begin{pmatrix} \theta \\ z \end{pmatrix} = \begin{pmatrix} I(i\omega)^2 & \theta \\ m(i\omega)^2 & z \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -\delta + I\omega^2 & -\frac{1}{2}\varepsilon \\ -\frac{1}{2}\varepsilon & -k+m\omega^2 \end{pmatrix}\begin{pmatrix} \theta \\ z \end{pmatrix} = 0$$

for non-trivial solutions the equations must be linearly dependent

$$\Rightarrow \det\left[\quad\right] = 0$$

$$\det \{ \quad \} = (-\delta + I\omega^2)(-k + m\omega^2) - \left(\frac{1}{2}\varepsilon\right)^2$$

$$= mI\omega^4 - (\delta m + Ik)\omega^2 + \delta k - \frac{1}{4}\varepsilon^2 = 0$$

$$\Rightarrow \quad \omega^4 - \frac{(\delta m + Ik)}{mI}\omega^2 + \frac{\delta k}{mI} - \frac{1}{4}\frac{\varepsilon^2}{mI} = 0$$

$$\Rightarrow \quad \omega^2 = \frac{1}{2}\frac{\delta m + Ik}{mI} \pm \sqrt{\left(\frac{1}{2}\frac{\delta m + Ik}{mI}\right)^2 - \frac{\delta k}{mI} + \frac{1}{4}\frac{\varepsilon^2}{mI}}$$

$$\left(= \frac{1}{2}\frac{\delta m + Ik}{mI} \pm \sqrt{\left(\frac{1}{2}\frac{\delta m - Ik}{mI}\right)^2 + \frac{1}{4}\frac{\varepsilon^2}{mI}} \right)$$

5 View φ as generalized coordinate and use (E)L for four integration variables

$$\frac{\partial \mathcal{L}}{\partial \varphi} = \sum_{\mu=0}^{3} \frac{\partial^{\text{tot}}}{\partial x^{\mu}} \underbrace{\left(\frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \varphi}{\partial x^{\mu}} \right)} \right)}_{\frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)}}$$

$$\frac{\partial \mathcal{L}}{\partial \varphi} = -m^2 \varphi - \lambda \varphi^3$$

$$\frac{\partial}{\partial x^{\mu}} \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \right) = \frac{\partial}{\partial x^{\mu}} \left(\frac{\partial}{\partial (\partial_\mu \varphi)} \frac{1}{2} g^{m\nu} \partial_m \varphi \, \partial_\nu \varphi \right)$$

$$= \frac{\partial}{\partial x^{\mu}} \left(\frac{1}{2} g^{m\nu} \left(\delta_m^{\mu} \partial_\nu \varphi + \partial_m \varphi \, \delta_\nu^{\mu} \right) \right)$$

$$= \partial_\mu \left(\frac{1}{2} g^{\mu\nu} \partial_\nu \varphi + \frac{1}{2} g^{m\mu} \partial_m \varphi \right)$$

$$= \partial_\mu \left(\frac{2 \partial^{\mu}}{2} \varphi \right) = \partial_\mu \partial^{\mu} \varphi$$

$$\Rightarrow \quad \underbrace{\partial_\mu \partial^{\mu}}_{\partial_m \partial^m} \varphi + m^2 \varphi + \lambda \varphi^3 = 0$$