Last time:



Assuming homogeneity, isotropy and that the speed of light is the same for all observers (and thus independent of the speed of the emitter) we derived the Lorentz transformations

$$\boxed{\begin{aligned} t' &= \gamma(u)\left(t - \frac{u}{c^2}x\right) \\ x' &= \gamma(u)\left(x - ut\right) \\ y' &= y \\ z' &= z \end{aligned}} \qquad \boxed{\gamma(u) = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}}$$

This is often written in matrix form

$$\boxed{\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \underbrace{\begin{bmatrix} \gamma & -\gamma\frac{u}{c^2} & 0 & 0 \\ -\gamma u & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\Lambda(u)} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}} \quad \begin{array}{l} \leftarrow \text{ component } 0 \\ \leftarrow \text{ component } 1 \\ \qquad\qquad\quad 2 \\ \qquad\qquad\quad 3 \end{array}$$

Lorentz transformation matrix

Remark: We have derived the Lorentz transformations by requiring

homogeneity, isotropy and that the speed of light is the same for all observers.

Most often the axioms of SR are formulated as

1) The laws of physics are the same for all observers in all inertial frames

2) All observers measure the same speed of light, c.

Note: It is enough that one observer measures c, because then, by 1), all observers measure c.

Note that 1) implies homogeneity and isotropy since if the universe was not homogeneous physics would depend on the observers location, and if it was not isotropic, there would be a preferred reference frame.

# Implications of the Lorentz transformations



We will now explore the consequences of the fact that Lorentz transformations (rather than Galilean transformations) are the transformations which leave the laws of physics invariant.

We have already seen:

$$\begin{cases} t' = t \\ x' = x - ut \\ y' = y \\ z' = z \end{cases}$$

* Lengths of objects in motion relative to you are contracted in the direction of the motion

* Time seems to pass slower for observers which move compared to you
  Clock will seem to go slower in the sense that it takes longer time between two events.

* As $t' = \gamma\left(t - \frac{u}{c^2}x\right)$ depends on x, events at same time t will have different t'.

As c is the maximal speed with which information propagates we also have

* Events separated in spacetime such that the distance in space is larger than the "distance in time", cΔt, are causally disconnected and cannot affect each other! Such events fulfill

  $$\sqrt{\Delta\bar{x}^2} > c\,|\Delta t| \quad \text{or} \quad (c\Delta t)^2 - (\Delta\bar{x})^2 < 0$$

  and have spacelike separation

* Events separated in time such that

  $$(c\Delta t)^2 - (\Delta\bar{x})^2 > 0$$

  are causally connected and are said to have timelike separation.

* Events with

  $$(c\Delta t)^2 - (\Delta\bar{x})^2 = 0$$

  have lightlike separation. Such events could be connected by light rays.

Remark: All observers must agree on whether two events are causally connected or not => they must agree on the sign of $(c\Delta t)^2 - (\Delta\bar{x})^2$

Check:

$t' = \gamma(t - \frac{u}{c^2}x)$
$x' = \gamma(x - ut)$
$y' = y$
$z' = z$

$$(c\Delta t')^2 - (\Delta x')^2 =$$

$$= c^2\gamma^2\left(\Delta t - \frac{u\Delta x}{c^2}\right)^2 - \gamma^2(\Delta x - u\Delta t)^2$$

$$= \gamma^2(c^2 - u^2)(\Delta t)^2 + 2\gamma^2\cancel{\left(-c^2\frac{u}{c^2} - -u\right)}\Delta x\Delta t + \gamma^2\left(\frac{u^2}{c^2} - 1\right)(\Delta x)^2$$

$\gamma^2 = \frac{1}{1 - \frac{u^2}{c^2}}$

$$= c^2(\Delta t)^2 - (\Delta x)^2$$

In fact all observers agree on the value of $c^2(\Delta t)^2 - (\Delta x)^2$

- $(\Delta s)^2 = c^2(\Delta t)^2 - (\Delta\bar{x})^2$ Remark: sign is a matter of convention

is the Lorentz invariant spacetime distance and a Lorentz scalar.

In Newtonian mechanics the laws of motion are invariant under Galilean transformations and spacetime is given by:

$$\underbrace{\mathbb{R}^3}_{\text{space}} \otimes \underbrace{\mathbb{R}}_{\text{time}}$$

The space part is equipped with a scalar product and a distance measure

$$(\Delta\bar{x})^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$ ← all observers agree on this in Newtonian mechanics

for each point in time separately.

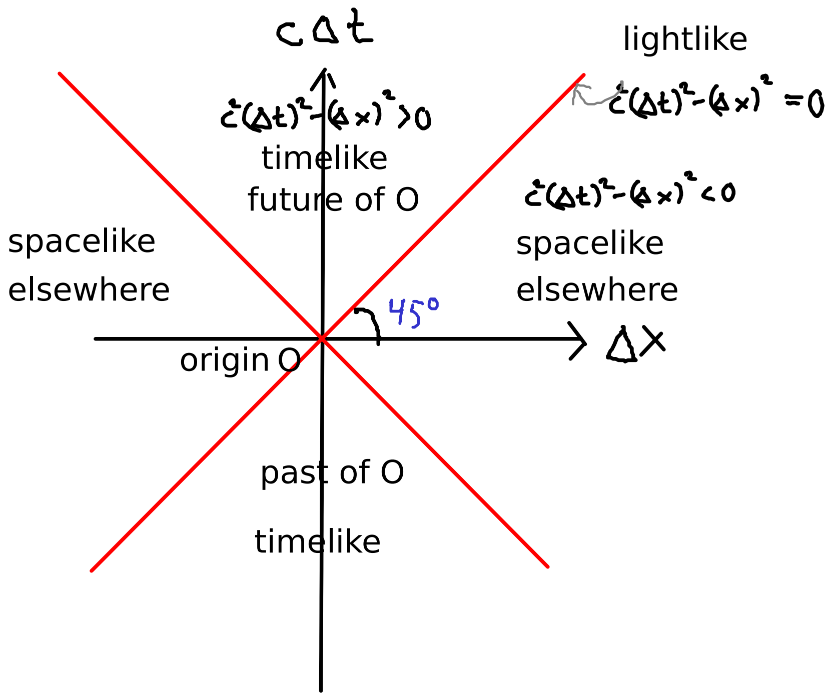
In special relativity the laws of physics are invariant under Lorentz transformations. Space and time are combined to a 4-dimensional spacetime known as Minkowski space, and the invariant distance measure is

$$\boxed{(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2}$$ ← all observers agree on this in special relativity

In fact, one could alternatively derive the Lorentz transformations by postulating

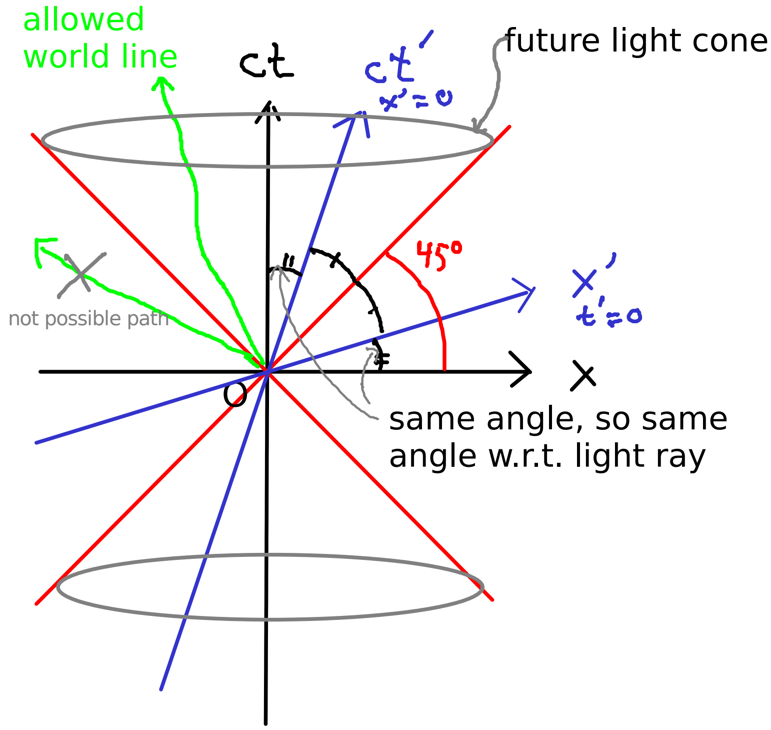
$(\Delta s)^2 = (c\Delta t)^2 - (\Delta\bar{x})^2$ to be invariant.

In a spacetime diagram or Minkowski diagram we have



Let

$$t' = \gamma\left(t - \frac{u}{c^2}x\right)$$
$$x' = \gamma\left(x - ut\right)$$




- For the t', axis x'=0

$$\Rightarrow\ 0 = \underbrace{\gamma(x - ut)}_{x'} \Rightarrow\ ct = \frac{c}{u}x, \quad \text{Ex: } u = \frac{1}{3}c \Rightarrow ct = 3x$$

- For the x', axis t'=0

$$\Rightarrow\ 0 = \underbrace{\gamma\left(t - \frac{ux}{c^2}\right)}_{t'} \Rightarrow\ ct = \frac{u}{c}x, \quad \text{Ex: } u = \frac{1}{3}c \Rightarrow ct = \frac{x}{3}$$

- For light in S', x'=ct'

$$\underbrace{\cancel{\gamma}(x - ut)}_{x'} = c\underbrace{\cancel{\gamma}\left(t - \frac{u}{c^2}x\right)}_{t'} \Rightarrow\ x\cancel{\left(1 + \frac{u}{c}\right)} = ct\cancel{\left(1 + \frac{u}{c}\right)} \Rightarrow\ x = ct$$

$\Rightarrow$ observers have same equation for light, as expected

## The "twin paradox"

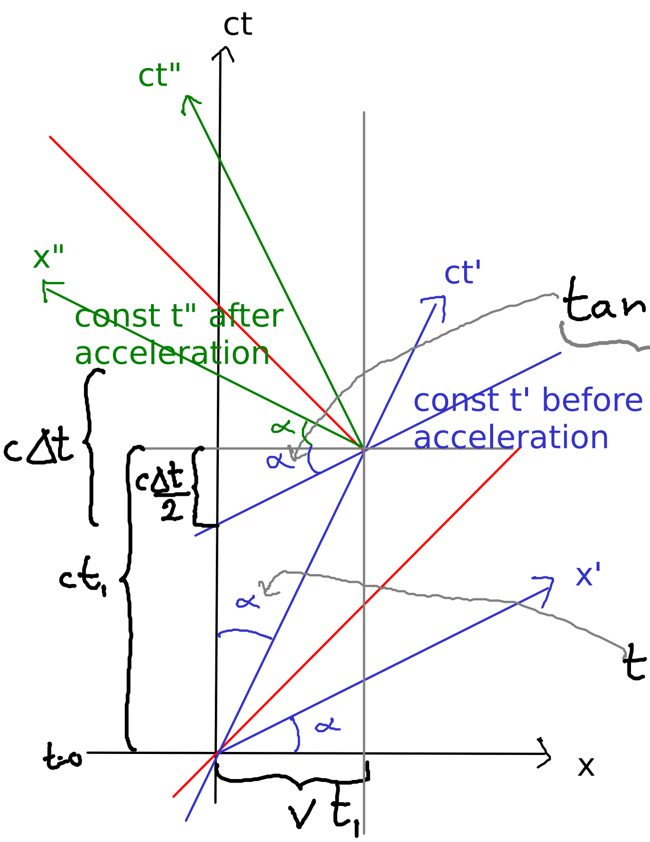
There are no paradoxes in SR, only situations that intuitively seem paradoxical!

The twin paradox is probably the most well known example. One twin stays at earth, whereas the other sets off at relativistic velocity v, travels for time $t_1$ as seen from the Earth, and then instantly changes velocity to -v and returns.

From the Earth's perspective the space traveling twin has a slow clock, and therefore will appeare to age slower. When the voyager comes back the earth twin thus expects to be younger than the astronaut.

But shouldn't the austronaut twin find the earth twin younger by the same argument?

This is the apparent paradox. In reality the two twins are different, as one undergoes an instant acceleration. In this moment something happens....




$$\underbrace{\tan\alpha}_{v/c} = \frac{c\Delta t/2}{v t_1} \Rightarrow \Delta t = 2\left(\frac{v}{c}\right)^2 t_1$$

$$\tan\alpha = \frac{v\cancel{t_1}}{c\cancel{t_1}}$$

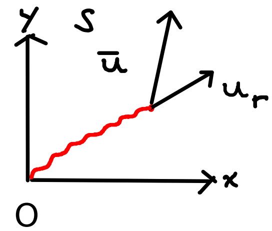
Ex: Let the distance be d=4 light years (in Earth's frame!), and v=0.8c then

- Time for earth twin: $2t_1 = \frac{2\cdot 4}{0.8} = 10$ years
- Time for astronaut: $\sqrt{1-\frac{v^2}{c^2}}\cdot 10 = 6$ years

Note: A rigorous treatment should involve acceleration!

# The relativistic Doppler effect

Let a light source P travel through the inertial frame S with relativistic velocity $\bar{u}$.
What frequency does an observer at O measure if the source has frequency $\nu_0$ in its own frame?




Let the time between two wave peak emissions be $\Delta t_0$ as measured in the frame of the source. In S the time between these emissions are then by time dilation $\Delta t = \frac{\Delta t_0}{\sqrt{1-\frac{\bar{u}^2}{c^2}}}$ (L1:3), but in this time the source has increased its distance by $\Delta t\, u_r$ so the wave peaks have a longer distance to travel.

extra distance to travel for ligth

In total it takes $\Delta t_{\text{peaks}} = \Delta t + \frac{\Delta t\, u_r}{c} = \frac{\Delta t_0}{\sqrt{1-\frac{\bar{u}^2}{c^2}}}\left(1 + \frac{u_r}{c}\right)$

speed of traveling extra distance

The wavelength is proportional to $\Delta t_{\text{peaks}}$

$\Rightarrow$ In total we have the relativistic Doppler effect

$$\boxed{\frac{\lambda}{\lambda_0} = \frac{\nu_0}{\nu} = \underbrace{\frac{1}{\sqrt{1-\frac{\bar{u}^2}{c^2}}}}_{\text{relativistic correction}}\underbrace{\left(1 + \frac{u_r}{c}\right)}_{\text{Galilean Doppler effect (for an observer in rest w.r.t. medium)}}}$$

In the special case that the source moves straight away from the observer:

$$\frac{\lambda}{\lambda_0} = \frac{\nu_0}{\nu} = \frac{1+\frac{u}{c}}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{1+\frac{u}{c}}{\sqrt{\left(1+\frac{u}{c}\right)\left(1-\frac{u}{c}\right)}} = \sqrt{\frac{1+\frac{u}{c}}{1-\frac{u}{c}}}$$

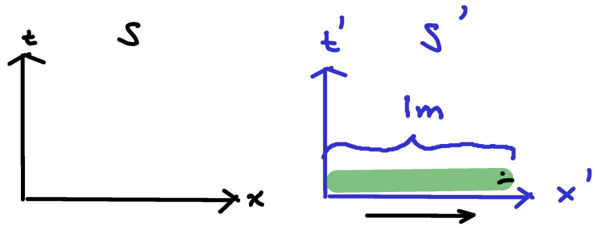
Comment: If $\frac{u}{c} \ll 1$ we get $\lambda = \lambda_0\left(1 + \frac{u}{c}\right)$

# The length contraction paradox

There are many versions of the paradox, the train in the tunnel, the ladder in the garage, the pole in the barn, etc.

Taylor: 613-615
Rindler: 26-27

Consider a relativistic snake of rest length 1m sliding over a table at speed v=0.6c.

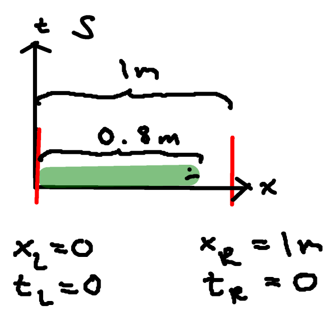



v=0.6c

In the table's system the snake is Lorentz contracted and has length:

$$\sqrt{1-\frac{v^2}{c^2}} = \sqrt{1-\left(\frac{0.6c}{c}\right)^2}\cdot 1 = \sqrt{1-0.36} = \sqrt{0.64} = 0.8\ m$$

A physics student wants to tease the snake by bouncing two cleavers with distance 1m, such that the snake is missed (see fig).




The snake thinks that the table and the distance between the cleavers is Lorentz contracted to length 0.8m, so it thinks it will be cut in pieces!

What is the problem?

Answer: the bounces are simultaneous in S, what about S'?

In S the bounces at $x_L$ and $x_R$ are simultaneous, but in S'

$$\begin{cases} x'_L = \gamma(v)(\underbrace{x_L}_{0} - v\underbrace{t_L}_{0}) = 0 \\ t'_L = \gamma(v)(\underbrace{t_L}_{0} - \frac{v}{c^2}\underbrace{x_L}_{0}) = 0 \end{cases} \qquad \begin{cases} x'_R = \gamma(v)(x_R - v\underbrace{t_R}_{0}) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\cdot 1m = \frac{1}{0.8}\cdot 1m = 1{,}25m \\ t'_R = \gamma(v)(\underbrace{t_R}_{0} - \frac{v}{c^2}x_R) = \overbrace{-1.25}^{\gamma(v)x_R}\cdot\frac{0.6}{c} = -\frac{0.75}{c} = -2.5ns \end{cases}$$

i.e. the bounce at $x'_R$ happens well before the bounce at $x'_L$. At t'=0, the mark in the table is at

$$x'_{\text{mark}}(t'=0) = x'_R + v_{\text{mark}}|t'_R| = 1{,}25 - 0.6c\,\frac{0.75}{c} = 0.8m$$

place at t'=-2.5 ns

so indeed the distance at equal t' is 0.8m.