Scalars four-vectors and tensors

Today's lecture may seem a bit dry, but I promise that it will be extremely important!

It turns out that it is extremely useful to classify physical entities according to how they transform i.e. change under Lorentz transformations.

In general the coordinates of an event can be written

$$(ct, x, y, z) = (x^0, x^1, x^2, x^3).$$

We may as well combine these coordinates to one object called a four-vector

$$\boxed{(x^0, x^1, x^2, x^3) = (ct, x, y, z)}$$

Here it is important that the indices sit upstairs!

This class of vectors is known as a contravariant four-vector.

As we have seen they transform (i.e. change under Lorentz transformations) with the Lorentz matrix

$$\boxed{\underbrace{\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix}}_{x'^\mu} = \underbrace{\begin{pmatrix} \gamma & -\gamma\frac{u}{c} & 0 & 0 \\ -\gamma\frac{u}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & \\ 0 & 0 & & 1 \end{pmatrix}}_{\Lambda(\frac{u}{c}) = \Lambda(\beta)} \underbrace{\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}}_{x^\nu}}$$

ct

We had:

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\frac{u}{c^2} & 0 & 0 \\ -\gamma u & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}$$

In component form this can be written

$$x'^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu x^\nu$$

$\Lambda^\mu{}_\nu$ = component $\mu\nu$ of Λ

Ex: $x'^0 = \Lambda^0{}_0 x^0 + \Lambda^0{}_1 x^1 + \Lambda^0{}_2 x^2 + \Lambda^0{}_3 x^3$

$$= \gamma x^0 - \gamma\frac{u}{c} x^1 + 0 \; x^2 + 0 \; x^3$$

Note: Λ is called p in Rindler

Note that the components of $\Lambda(\frac{u}{c})$ can be written as derivatives of primed coordinates w.r.t. unprimed

$$\boxed{\Lambda^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu}}$$

(= const)

Remark: Note that the Lorentz matrix is symmetric now.

Remark: It looks even simpler in natural units where c=1.

Remark: In general relativity the coefficients $\Lambda^{\mu}{}_{\nu}$ depend on the coordinates.

In fact, the concept contravariant four-vector is defined like this:

Def: a contravariant 4-vector is an object transforming like

$$\boxed{a'^{\mu} = \sum_{\nu} \frac{\partial x'^{\mu}}{\partial x^{\nu}} a^{\nu}}$$

The "simplest" objects - the scalars - don't change at all when they are Lorentz boosted. We have already encountered one such object, the invariant spacetime distance

$$(\Delta s)^2 = c^2(\Delta t)^2 - (\Delta \bar{x})^2 = c^2(\Delta t')^2 - (\Delta \bar{x}')^2 = (\Delta s')^2 .$$

In the new 4-vector notation this may be written as

$$(\Delta s)^2 = (c\Delta t)^2 - (\Delta \bar{x})^2 = (\Delta x^0)^2 - (\Delta x^1)^2 - (\Delta x^2)^2 - (\Delta x^3)^2$$

or

$$(\Delta s)^2 = (\Delta x^0, \Delta x^1, \Delta x^2, \Delta x^3) \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}}_{g} \begin{pmatrix} \Delta x^0 \\ \Delta x^1 \\ \Delta x^2 \\ \Delta x^3 \end{pmatrix}$$

or

$$(\Delta s)^2 = \sum_{\substack{\mu=0,1,2,3 \\ \nu=0,1,2,3}} \Delta x^{\mu} g_{\mu\nu} \Delta x^{\nu}$$

where g is known as the metric, or the covariant metric and its components are:

$$\boxed{\begin{array}{ll} g_{00} = 1 & g_{ij} = -\delta_{ij} \\ g_{0i} = g_{i0} = 0 & i,j = 1,2,3 \end{array}}$$

where δ_{ij} is the Kronecker delta $\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

Remark: Generally we let greek indices run from 0 to 3, and latin run from 1 to 3.

Remark: In general relativity the metric is a symmetric spacetime dependent matrix which contains the information about how the spacetime is curved.

Remark: In principle one may argue that the coordinates x^μ are no real vectors as their values depend on where we place the origin, such that only differences in coordinates are real vectors. As this rarely leads to confusion we ignore this distinction.

Writing $s^2 = \sum_{\mu\nu} x^\mu g_{\mu\nu} x^\nu$ is almost as clumsy as writing

$x^{0\,2} - x^{1\,2} - x^{2\,2} - x^{3\,2}$

We therefore introduce the Einstein summation convention where indices that appear twice are implicitly summed over. We can then write

$$s^2 = \sum_{\substack{\mu=0,1,2,3\\ \nu=0,1,2,3}} x^\mu g_{\mu\nu} x^\nu = x^\mu \underbrace{g_{\mu\nu} x^\nu}_{x_\mu}$$

Introducing the covariant four-vector

$$x_\mu = g_{\mu\nu} x^\nu \Leftrightarrow (x_0, x_1, x_2, x_3) = (ct, -x, -y, -z)$$

where the index always is placed downstairs, we may write s^2 in an even more elegant form

$$s^2 = x^\mu x_\mu$$

where it's again implicit that we sum over the index μ .

Remark: The quantity

$$\overset{\text{all 4 components of x and y}}{x \cdot y} = x^\mu y_\mu$$

is referred to as the scalar product between x and y, although this is not mathematically correct, as it's not positive semidefinite for $x \cdot x$.

An example of a covariant vector is the four-gradient $\frac{\partial f}{\partial x^{\mu}}$

Consider now

(by chain rule)

$$\frac{\partial f}{\partial x'^{\mu}} = \frac{\partial f}{\partial x^{\nu}} \frac{\partial x^{\nu}}{\partial x'^{\mu}}$$

As contravariant 4-vectors, the covariant 4-vectors are really defined by how they transform. From this we see that a covariant 4-vector transforms as

$$\boxed{b'_{\mu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} b_{\nu}}$$

Remark: In general an upper index inside a derivative "works" i.e. transforms as a lower index. Similarly a lower index in a derivative "works" as an upper index.

We can now check that the scalar product between a covariant and a contravariant 4-vector is indeed a scalar (i.e. doesn't change under Lorentz boosts)

$$a'^{\mu} b'_{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} a^{\nu} \frac{\partial x^{\rho}}{\partial x'^{\mu}} b_{\rho} = \underbrace{\frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial x'^{\mu}}{\partial x^{\nu}}}_{\text{by inverse chain rule}} a^{\nu} b_{\rho} =$$

Recall: Implicit sum over repeated indices!

$$\frac{\partial x^{\rho}}{\partial x^{\nu}} = \delta^{\rho}_{\nu}$$

$$= \underbrace{\delta^{\rho}_{\nu} a^{\nu}}_{a^{\rho}} b_{\rho} = a^{\rho} b_{\rho}$$

OK

Contravariant and covariant four-vectors thus transform in compensating ways.

In general we may have yet more complicated objects which transform in more complicated ways when Lorentz boosted. Such objects are generally called tensors.

So far we have encountered four types of tensors

1) Scalars which do not change at all. They carry no free indices and are said to have rank 0 or more carefully $\binom{0}{0}$ 0 indices upstairs
0 indices downstairs

2) Contravariant 4-vectors which transform as $a'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} a^{\nu}$
and have one upper index are said to have rank 1 or more carefully
$\binom{1}{0}$ 1 index upstairs
0 indices downstairs

3) Covariant 4-vectors which transform as $b'_{\mu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} b_{\nu}$
and have one lower index are said to have rank 1, or more carefully
$\binom{0}{1}$ 0 indices upstairs
1 index downstairs

4) The covariant metric $g_{\mu\nu}$ which has 2 lower indices and rank 2,
or more carefully $\binom{0}{2}$ 0 indices upstairs
2 indices downstairs

Let's see how g transforms, recall that

$$x^{\mu} x_{\mu} = x^{\mu} g_{\mu\nu} x^{\nu}$$

is a scalar and thus must remain invariant, giving:

express unprimed indices in primed indices, hence the unprimed sit upstairs despite that the vector is contravariant (transform backwards)

$$s'^2 = x'^{\mu} g'_{\mu\nu} x'^{\nu}$$
$$\parallel$$

want primed version of g

$$s^2 = g_{\rho\sigma} x^{\rho} x^{\sigma} = g_{\rho\sigma} \left(\frac{\partial x^{\rho}}{\partial x'^{\mu}} x'^{\mu} \right) \left(\frac{\partial x^{\sigma}}{\partial x'^{\nu}} x'^{\nu} \right)$$

$$\Rightarrow \quad g'_{\mu\nu} = g_{\rho\sigma} \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}}$$

as non-blue must agree

Note: We did not use that the metric is a special tensor for the transformation.

In general, by definition, a rank $\binom{0}{2}$ tensor transforms like this

$$T'_{\mu\nu} = T_{\rho\sigma}\,\frac{\partial x^\rho}{\partial x'^\mu}\underbrace{\frac{\partial x^\sigma}{\partial x'^\nu}}_{\text{One factor like this for each lower index}} .$$

Similarly a rank $\binom{2}{0}$ tensor transforms as

$$T'^{\mu\nu} = T^{\rho\sigma}\,\frac{\partial x'^\mu}{\partial x^\rho}\underbrace{\frac{\partial x'^\nu}{\partial x^\sigma}}_{\text{One factor like this for each upper index}} ,$$

and a mixed tensor as

$$T'^{\mu}{}_{\nu} = T^{\rho}{}_{\sigma}\,\frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x^\sigma}{\partial x'^\nu} .$$

But the tensor concept is more general than this. We may have any number of lower and upper indices, and a general rank $\binom{m}{n}$ tensor transforms as

$$\boxed{T'^{\overbrace{\mu\cdots\nu}^{\text{m indices}}}{}_{\underbrace{\alpha\cdots\beta}_{\text{n indices}}} = T^{\tilde\mu\cdots\tilde\nu}{}_{\tilde\alpha\cdots\tilde\beta}\left(\frac{\partial x'^\mu}{\partial x^{\tilde\mu}}\cdots\frac{\partial x'^\nu}{\partial x^{\tilde\nu}}\right)\left(\frac{\partial x^{\tilde\alpha}}{\partial x'^\alpha}\cdots\frac{\partial x^{\tilde\beta}}{\partial x'^\beta}\right)}$$

As we have seen the (covariant) metric $g_{\mu\nu}$ has the effect of lowering indices:

$$\boxed{g_{\mu\nu}x^{\nu} = x_{\mu}}$$

Its inverse is known as the contravariant metric, and has the effect of raising indices

$$\boxed{g^{\mu\nu}x_{\nu} = x^{\mu}}$$

$g^{\mu\nu}$ is, as the notation suggests, a rank $\binom{2}{0}$ tensor.

Note that we always place indices such that they indicate how the tensor transforms

We also introduce some useful derivative operators

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- covariant derivative
  transforms covariantly

$$\boxed{\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right) = \left(\frac{\partial}{\partial x^0}, \vec{\nabla}\right)}$$

- contravariant derivative
  transforms contravariantly

$$\boxed{\partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right) = \left(\frac{\partial}{\partial x^0}, -\vec{\nabla}\right)}$$

- d'Alemberts operator
  scalar

$$\boxed{\Box = \partial^\mu \partial_\mu = \frac{\partial}{\partial x_0}\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x_i}\frac{\partial}{\partial x^i} = \frac{\partial^2}{(\partial x^0)^2} - \vec{\nabla}^2}$$

$x^0 = x_0$, $x_i = -x^i$
as
$g_{00} = 0$, $g_{ij} = -\delta_{ij}$

- 4-divergence
  scalar

$$\boxed{\partial^\mu a_\mu}$$

For a 4-vector square we have

$$a \cdot a = a^{\mu} a_{\mu} = (a^0)^2 - (\bar{a})^2$$

thus

$a^2 < 0$ spacelike

$a^2 > 0$ timelike

$a^2 = 0$ lightlike