Basic four-momentum kinematics

Rindler: Ch5: sec. 25-30, 32

HUB, (II.6-)II.7, p142-146
+part of I.9-1.10, 154-162

Last time we intruduced the contravariant 4-vector

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$$

vector ← The world is inconsistent! → component

and the covariant 4-vector

$$x_\mu = g_{\mu\nu} x^\nu = (ct, -x, -y, -z)$$

implicit sum over ν

$$\left(\text{as} \quad \begin{array}{l} g_{00}=1, \; g_{ij} = -\delta_{ij} \\ g_{i0} = g_{0i} = 0 \end{array} \right)$$

We also introduced the scalar product

$$a \cdot b = a^\mu b_\mu = a^0 b_0 + a^1 b_1 + a^2 b_2 + a^3 b_3$$

$$= a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3$$

For a 4-vector square we have

$$a \cdot a = a^\mu a_\mu = (a^0)^2 - (\bar{a})^2$$

thus

$a^2 < 0$ spacelike

$a^2 > 0$ timelike

$a^2 = 0$ lightlike

Today we will introduce some useful 4-vectors, but first we introduce the

proper time, $\Delta\tau$ which is simply the time percieved in an intertial frame
(i.e. time by a clock moving with observer)

If the observer is at rest, then only the time component changes

$$\Delta\tau = \sqrt{(\Delta t)^2} = \sqrt{\frac{(\Delta s)^2}{c^2}} = \frac{1}{c}\sqrt{(\Delta s)^2}$$

but all observers agree on ΔS, therefore we have for an observer at

constant speed

$$\boxed{\Delta\tau = \frac{1}{c}\sqrt{c^2(\Delta t)^2 - (\Delta \bar{x})^2} = \frac{1}{c}\Delta t\sqrt{c^2 - \underbrace{\left(\frac{\Delta \bar{x}}{\Delta t}\right)^2}_{\bar{v}^2}} = \Delta t\sqrt{1 - \frac{\bar{v}^2}{c^2}}}$$

For a general world line, corresponding to an accelerating observer, we have

$$\boxed{\tau = \int dt \sqrt{1-\frac{\bar{v}^2(t)}{c^2}} = \int \frac{dt}{\gamma(\bar{v}(t))}}$$

Using this it makes sense to define the 4-velocity

$$\boxed{u^\mu \equiv \frac{dx^\mu}{d\tau}}$$

As x^μ transforms as a contravariant 4-vector and τ as a scalar u^μ indeed transforms as a contravariant 4-vector, so the notation makes sense!

We also introduce the 4-acceleration $a^\mu = \frac{du^\mu}{d\tau} = \frac{d^2x^\mu}{d\tau^2}$

Let's calculate the 4-velocity:

$$u^\mu = \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt}\frac{dt}{d\tau} = \frac{dx^\mu}{dt}\gamma = \gamma \frac{d}{dt}(ct, \bar{x})$$

$$= \gamma(c, \bar{v}) = \gamma c\left(1, \frac{\bar{v}}{c}\right) = \gamma c(1, \bar{\beta})$$

and the 4-velocity square

$$u^\mu u_\mu = u^\mu g_{\mu\nu} u^\nu = (u^0)^2 - (\bar{u})^2 =$$

$$= \gamma^2(1-\bar{\beta}^2)c^2 = \left(\frac{1}{\sqrt{1-\bar{\beta}^2}}\right)^2 (1-\bar{\beta}^2)c^2 = c^2 \qquad (1)$$

Multiplying the 4-velocity with the mass we get the 4-momentum

$$\boxed{p^\mu = m u^\mu = m\gamma(c, \bar{v})} \qquad (2)$$

Note: In Rindler m is called m_0 and Rindler's $m = m_0\gamma$ I will always mean m_0 with m .

which transforms as, i.e. is, a contravariant 4-vector.

Remark: in some (old) literature the factor

$$\boxed{m_{rel} = m\gamma}$$

($m = m_0$)

is referred to as the relativistic mass or relativistic inertial mass.

The spatial components of the 4-momentum is the relativistic 3-momentum

or simply relativistic momentum

$$\boxed{\bar{p} \overset{(2)}{=} m_0 \gamma \bar{v}} \Rightarrow \bar{p}^2 = m_0^2 \gamma^2 \bar{v}^2$$

and the 0-component turns out to give the energy:

$$\boxed{E = cp^0 \overset{(2)}{=} c\, m \gamma c} \Rightarrow \boxed{E^2 = m^2 \gamma^2 c^4}$$

Remark: Taylor expanding for small v we get:

- $$E = p^0 c = m \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} c^2 = mc^2\left(1 + \frac{1}{2}\left(\frac{v}{c}\right)^2 + \dots\right) = \left(mc^2 + \frac{1}{2} m \bar{v}^2 + \dots\right)$$

  rest energy nonrelativistic kinetic energy

  for v=0 $E = mc^2$

- $$\bar{p} = m\gamma\bar{v} = \underbrace{m\bar{v}}_{\text{nonrelativistic momentum}} + \dots$$

For the 4-momentum square we have:

$$\boxed{p^\mu p_\mu = m^2 u^\mu u_\mu \overset{(1)}{=} m^2 c^2}$$

$$p^\mu p_\mu = \underbrace{m^2 \gamma^2 c^2}_{E^2/c^2} - \underbrace{m^2 \gamma^2 \bar{v}^2}_{\bar{p}^2} \Rightarrow \boxed{E^2 = m^2 c^4 + \bar{p}^2 c^2}$$

As you may expect we have conservation of 4-momentum, i.e. summing over

all incoming or outgoing particles (including photons)

$$\boxed{P^\mu = \sum_{\substack{i,\text{incoming} \\ \text{particle}}} p_i^\mu = \sum_{\substack{o,\text{ outgoing} \\ \text{particle}}} p_o^\mu}$$

The square of P^μ, $P^\mu P_\mu$ is c^2 times the invariant mass square, $P^\mu P_\mu = M^2 c^2$

M^2 is a very useful quantity as it is both conserved and Lorentz invariant!

In particular, for a single particle we have

$$p^\mu p_\mu = p^\mu p_\mu = m^2\gamma^2(c,\bar{v})^2 = m^2\frac{1}{1-\frac{\bar{v}^2}{c^2}}(c^2-\bar{v}^2) = m^2c^2$$

so the name makes sense

Note that since the 4-momentum is a 4-vector it transforms as a 4-vector,

i.e. with the Lorentz matrix

$$\boxed{\begin{bmatrix} p'^0 \\ p'^1 \\ p'^2 \\ p'^3 \end{bmatrix} = \begin{bmatrix} \gamma & -\frac{u}{c}\gamma & 0 & 0 \\ -\frac{u}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p^0 \\ p^1 \\ p^2 \\ p^3 \end{bmatrix}} \begin{array}{l} \leftarrow E/c \\ \leftarrow p^x \\ \leftarrow p^y \\ \leftarrow p^z \end{array}$$

We note that there is a particular Lorentz frame in which calculatioins become easy, namely the center of momentum frame, "the CM-frame" in which

$$\sum_{\text{particles}} \bar{p}_{\text{particle}} = \bar{0}$$

Don't confuse this with the center of mass system in which the origin is at

$$\bar{r}_{CM} = \frac{\sum_i m_i \bar{r}_i}{\sum_i m_i}.$$

For the center of momentum frame it does not matter where we place the origin as only the motion of the CM-frame matters.

Boosting to the center of momentum system - the general case

As mentioned, calculations get particularly easy in the CM-system. We may

therefore want to boost to and from the CM-system when calculating.

In the center of momentum system the total spatial momentum is $\bar{0}$.

If we are in a system with total momentum $\bar{p}$

we know that to "get rid of it" we have to boost in direction $\bar{p}$

But what is the magnitude of the boost?

Assume for a while that $\bar{p} = (p^1, 0, 0)$

and write down a Lorentz boost, requiring the spatial momentum to disappear:

$$\begin{bmatrix} E_{CM}/c \\ 0 \\ 0 \\ 0 \end{bmatrix} \overset{!}{=} \begin{bmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} E_{lab}/c \\ p^1_{lab} \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow p^1_{CM} \overset{!}{=} 0 = -\cancel{\gamma}\beta(E_{lab}/c) + \cancel{\gamma} p^1_{lab}$$

$$\Rightarrow \beta = \frac{p^1_{lab}\, c}{E_{lab}}$$

From this we conclude that for a general momentum direction we have:

$$\boxed{\bar{\beta} = \frac{\bar{p}_{lab}\, c}{E_{lab}}}$$

(This corresponds to 30.2 in Rindler.)

Photons

For photons we have:

$$\lambda = \frac{h}{|p|}$$

$$E = \hbar\omega = h\nu = \frac{hc}{\lambda} = c|p|$$

$$e^{-i(\omega t - \bar{k}\cdot\bar{r})} = e^{-i(\underbrace{\frac{\omega}{c}}_{k^0}\underbrace{tc}_{x^0 = x_0} - \underbrace{\bar{k}\cdot\bar{r}}_{\sum k^i r^i})} \overset{r^i = -r_i}{=} e^{-i(k^\mu x_\mu)}$$

where we have introduced the wave vector: $k^\mu = (\frac{\omega}{c}, \bar{k}) = \frac{\omega}{c}(1, \hat{e})$ unit vector

giving us lightlike photons:

$$\boxed{k^\mu k_\mu = \left(\frac{\omega}{c}\right)^2 (1^2 - \hat{e}^2) = 0}$$

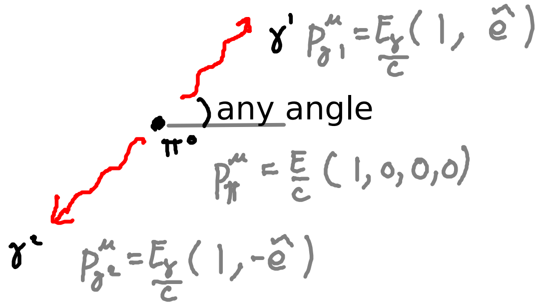
Multiplying with $\hbar$ gives the 4-momentum of a photon:

$$\boxed{p^\mu = \hbar k^\mu = \hbar\frac{\omega}{c}(1, \hat{e})}$$

Example:

Consider a pion decaying in its restframe (the CM-frame) to 2 photons

We know that the 4-momentum is conserved:



$$p^\mu_\pi = \frac{E}{c}(1,0,0,0) = p^\mu_{\gamma 1} + p^\mu_{\gamma 2}$$

As the total spatial momentum is 0 before the decay we know that the photons have equally large and compensating spatial momenta:

$$\bar{p}_{\gamma 1} = -\bar{p}_{\gamma 2}$$

but, for photons

$$p^\mu_\gamma p_{\gamma\mu} = 0 = \left(\frac{E_\gamma}{c}\right)^2 - \bar{p}_\gamma^{\,2} \Rightarrow \frac{E_\gamma}{c} = |\bar{p}_\gamma|$$

this implies that the photons have equal energy.

From conservation of the 0th component in the 4-vector, i.e. energy conservation, we have:

$$p^0_\pi = \frac{E_\pi}{c} = \frac{m_\pi c^2}{c} = \frac{E_{\gamma 1} + E_{\gamma 2}}{c} = \frac{2}{c} E_\gamma$$

$$\Rightarrow \quad E_\gamma = \frac{1}{2} E_\pi = \frac{1}{2} m_\pi c^2$$

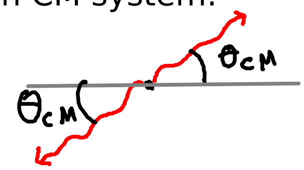
$$\Rightarrow \quad |\bar{p}_\gamma| = \frac{1}{2} m_\pi c$$

.

Assume now that the pion system is boosted with a factor β in the z-direction.

In the CM-system the photons go out back to back, but with some angle compared to the boost axis. Taking their 3-momenta to be in the xz-plane we have:

In CM system:



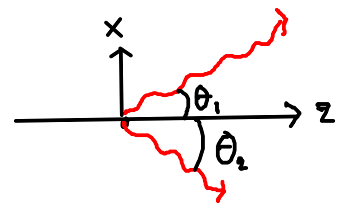
$$\hat{p}_{\gamma 1,2\,CM} = \frac{1}{2} m_\pi c \left(1, \pm \sin\theta_{CM}, 0, \pm\cos\theta_{CM}\right)$$

To see what it looks like in the lab-system we may boost us back to the lab with $\bar{\beta} = (0,0,-\beta)$ giving

$$\hat{p}_{\gamma 1,2\,lab} = \begin{bmatrix} \gamma & 0 & 0 & -\gamma(-\beta) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma(-\beta) & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} 1 \\ \pm\sin\theta_{CM} \\ 0 \\ \pm\cos\theta_{CM} \end{bmatrix} \frac{1}{2} m_\pi c$$

$$= \frac{m_\pi c}{2} \begin{bmatrix} \gamma + \gamma\beta(\pm\cos\theta_{CM}) \\ \pm\sin\theta_{CM} \\ 0 \\ \gamma\beta + \gamma(\pm\cos\theta_{CM}) \end{bmatrix}$$

In lab:



In the lab system we have

$$\tan\theta_{1,2,lab} = \frac{p^x_{\gamma 1,2,lab}}{p^z_{\gamma 1,2,lab}} = \frac{\pm\sin\theta_{CM}}{\gamma(\beta \pm \cos\theta_{CM})}$$

$$= \pm \underbrace{\sqrt{1-\beta^2}}_{\beta\to 1 \;\to\; 0} \frac{\overbrace{\sin\theta_{CM}}^{>0}}{\underbrace{(\beta \pm \cos\theta_{CM})}_{>0 \text{ for } \beta\to 1}} \xrightarrow{\beta\to 1} \pm 0$$

There cannot be a preferred direction for the pion decay in its rest system. However, when $\beta \to 1$ we see that both photons move in the positive z-direction in the lab.

Example: Compton scattering

A photon is scattered an angle θ_γ by an electron initially at rest.

What is the change in wavelegth of the photon?

Before:

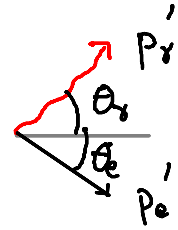
We know that the total 4-momentum is conserved:

$$p_e^{\mu} + p_\gamma^{\mu} = p_e'^{\mu} + p_\gamma'^{\mu}$$

$$\Leftrightarrow p_e'^{\mu} = p_e^{\mu} + p_\gamma^{\mu} - p_\gamma'^{\mu}$$

[by writing p_e' alone we have no θ_e dependence]

After:



Squaring gives:

$$\Rightarrow \cancel{\underbrace{p_e'^2}_{m^2c^2}} = \cancel{\underbrace{p_e^2}_{m^2c^2}} + \underbrace{p_\gamma^2}_{0} + \underbrace{(-p_\gamma')^2}_{0} + 2p_e\cdot p_\gamma - 2p_e\cdot p_\gamma' - 2p_\gamma\cdot p_\gamma'$$

(m is the electron mass)

$$\Leftrightarrow p_\gamma\cdot p_\gamma' = p_e\cdot p_\gamma - p_e\cdot p_\gamma'$$

only 0-component is non-zero at rest

$$\Leftrightarrow \frac{E_\gamma E_\gamma'}{c^2} - \underbrace{\bar{p}_\gamma\cdot \bar{p}_\gamma'}_{\frac{1}{c^2}E_\gamma E_\gamma'\cos\theta_\gamma} = mc\frac{E_\gamma}{c} - mc\frac{E_\gamma'}{c}$$

$$\Leftrightarrow \frac{E_\gamma E_\gamma'}{c^2}(1-\cos\theta_\gamma) = m(E_\gamma - E_\gamma')$$

Relating the photon wavelegth to its energy we have: $E = \frac{hc}{\lambda}$

$$\frac{1}{c^2}\frac{\cancel{hc}}{\cancel{\lambda}}\frac{hc}{\cancel{\lambda'}}(1-\cos\theta_\gamma) = m\cancel{hc}\underbrace{\left(\frac{1}{\lambda}-\frac{1}{\lambda'}\right)}_{\frac{\lambda'-\lambda}{\cancel{\lambda\lambda'}}}$$

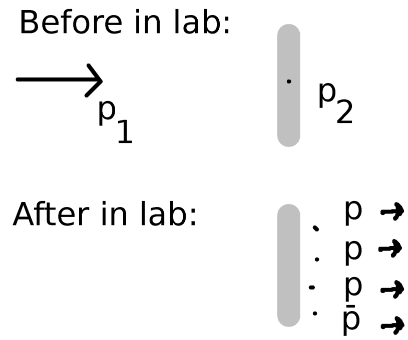
$$\Rightarrow \lambda'-\lambda = \frac{h}{mc}(1-\cos\theta_\gamma)$$

Example:

Assume we want to create anti-protons by shooting protons on a fixed target,

p+p-> p+p+$\bar{p}$+p.

What is the minimal energy of the proton p_1 ?



In the lab-frame, for motion in z-direction, we have:

$$P^\mu = p_1^\mu + p_2^\mu = (\gamma m_p c, 0, 0, \gamma m_p v) + (c\, m_p, 0, 0, 0)$$
$$= m_p c\left((\gamma+1), 0, 0, \gamma \frac{v}{c}\right)$$

giving for the invariant mass

$$M^2 = \frac{P^\mu P_\mu}{c^2} = m_p^2\left((\gamma+1)^2 - 0^2 - 0^2 - \gamma^2 \frac{v^2}{c^2}\right)$$

But this quantity is conserved and the same in all systmes (i.e. a Lorentz scalar)

so, in the CM system where the momentum is 0 it must equal the (total mass)^2

$$\Rightarrow \cancel{m_p^2}\left(\gamma^2 + 2\gamma + 1 - \gamma^2 \frac{v^2}{c^2}\right) \stackrel{!}{=} (4\cancel{m_p})^2$$
$$\Leftrightarrow \underbrace{\gamma^2\left(1 - \frac{v^2}{c^2}\right)}_{1} + 2\gamma + 1 = 4^2$$
$$\Leftrightarrow 2\gamma + 2 = 16$$
$$\Leftrightarrow \gamma = 7$$

From the 0-component, we find the energy $E_{p_1} = \gamma m_p c^2 = 7 m_p c^2$

i.e., the kinetic energy is $6 m_p c^2$. This is larger than the energy needed to create

2 proton masses.

This is why both particles are accelerated in collider experiments, such as LHC!

The lowest energy for which a reaction can take place is called the threshold energy.

Collisions with different particle content on the incoming and outgoing side are

called inelastic, whereas collisions with the same particles are called elastic.