Forces in special relativity

Rindler: 90-92

Analogous to the non-relativistic force relation

$$\bar{f}_{\text{non-rel}} = m\bar{a} = \frac{d}{dt}(m\bar{v}) = \frac{d}{dt}(\bar{p}) \qquad \text{(Non-relativistic)}$$

we introduce the four-force

$$F = \frac{dP}{d\tau} \underbrace{\left(= \frac{d}{d\tau}(m_0 u)\right)}_{\text{If } m_0 \text{ is constant, generally } m_0 \text{ may change as energy} \sim \text{mass}} = \frac{d}{d\tau}\left(\frac{E}{c}, \bar{P}\right)$$

(Like Rindler I here let capital letters without indices denote contravariant four-vectors.)

where P is the contravariant four-momentum.

Since P is a four-vector and τ is a scalar F is a four-vector.

Using

$$\frac{d}{d\tau} = \frac{dt}{d\tau}\frac{d}{dt} = \gamma(\bar{v})\frac{d}{dt}$$

introduce $\bar{f} = \frac{d\bar{P}}{dt}$

we get $\quad P = \gamma(\bar{v})\, m(c, \bar{v})$

$$F = \gamma(\bar{v})\frac{d}{dt}P = \gamma(\bar{v})\frac{d}{dt}(\gamma(\bar{v})m_0 c, \bar{P}) = \gamma(\bar{v})\left(\frac{1}{c}\frac{dE}{dt}, \bar{f}\right)$$

Consider

four-velocity

$$F \cdot U = \gamma(\bar{v})\left(\frac{1}{c}\frac{dE}{dt}, \bar{f}\right) \cdot \gamma(\bar{v})(c, \bar{v})$$

$$= \gamma(\bar{v})^2\left(\frac{dE}{dt} - \bar{f}\cdot\bar{v}\right) = c^2\frac{dm_0}{d\tau}$$

This is a scalar => same in all systems
=> consider in rest system in which $E = mc^2,\ \bar{v} = 0,\ \gamma = 1$

From this we see

$$\frac{d}{d\tau}m_0 = 0 \Leftrightarrow \frac{d}{dt}m_0 = 0 \Leftrightarrow F\cdot U = 0$$

(A force for which m does not change is called pure in Rindler)

i.e., the (rest) mass is conserved iff $F \cdot U = 0$

Relativistic structure of Maxwell's theory

Rindler:
VI: 102-108

In fact, as remarked before, Maxwell's equations are already invariant under Lorentz transformations. Today we will assume we do not know them and ask the question:

What form can a relativistic electromagnetic force possibly have?

Assume:

- It preserves the rest mass => $F \cdot U \overset{!}{=} 0$

  (expected of electromagnetic force, but not in quantum field theories, where particles are created)

- It is proportional to the charge

1) Assume (incorrectly) that the four-force $F = (F^0, F^1, F^2, F^3)$ does not depend on the velocity, but then it cannot be a four-vector, since it cannot transform like one.

(The force can thus only be velocity indepndent in one frame.)

2) Next assume $F^\mu \propto U^\mu$ but then $F \cdot U \propto U \cdot U = c^2 \neq 0$

$\Rightarrow$ The force cannot preserve (rest) mass.

3) The third simplest guess is that the force depends linearly on the four-velocity:

$$F_\mu \left(= g_{\mu\nu} F^\nu\right) = \frac{q}{c} F_{\mu\nu} U^\nu$$

covariant four-vector

factor c for convenience

four-velocity, contravariant four-vector

(The tensor $F_{\mu\nu}$ is called $F_{\mu\nu}$ almost everywhere, but Rindler calls it $E_{\mu\nu}$)

must transform as a rank $\binom{0}{2}$ tensor

where $F_{\mu\nu}$ is the electromagnetic field-tensor.

Next we require that the rest mass doesn't change $\Rightarrow F\cdot U = 0$

$$F\cdot U = F_\mu U^\mu = \frac{q}{c} F_{\mu\nu} \underbrace{U^\nu U^\mu}_{\text{symmetric in } \mu \leftrightarrow \nu} \stackrel{!}{=} 0$$

so we see that if $F_{\mu\nu}$ is antisymmetric in $\mu \leftrightarrow \nu$ i.e., $F_{\mu\nu} = -F_{\nu\mu}$

then $F\cdot U = 0 \Rightarrow$ rest mass conserved.

Ex: The sum contains the 13 and 31-terms:

$$F_{13} U^1 U^3 + F_{31} U^3 U^1 = \underbrace{(F_{13} + F_{31})}_{0 \text{ if } F_{31} = -F_{13}} U^1 U^3$$

We have assumed that the fields affect the charges in proportion to U,

let's assume that the charges affect the fields in proportion to U

$$\Rightarrow \partial_\mu F^{\mu\nu} \underset{\text{def of } ,\mu}{=} F^{\mu\nu}{}_{,\mu} = k\rho_0 U^\nu \underset{\text{def of J}}{=} kJ^\nu \quad (*)$$

(We can raise and lower indices as we want with $g^{\mu\nu}$)

where we differentiate w.r.t. all indices standing after ,

- ρ_0 is the proper charge density, the density in the rest system
- J^μ is a four-vector defined by this equation, the four-current

From $kJ^\nu{}_{,\nu} \stackrel{(*)}{=} F^{\mu\nu}{}_{,\mu\nu} = -F^{\nu\mu}{}_{,\mu\nu} = -F^{\nu\mu}{}_{,\nu\mu} = -kJ^\nu{}_{,\nu}$

differ only by dummy indices

we conclude $\boxed{J^\nu{}_{,\nu} = 0}$ the continuity equation

Cf. $\frac{\partial \rho}{\partial t} + \nabla\cdot\bar{j} = 0$

Note that $F^{\mu\nu}$ has $\frac{4^2 - 4}{2} = 6$ components

diagonal entries are 0

antisymmetry

but we only have 4 field equations $(*)$

$\Rightarrow$ Not enough information to constrain $F^{\mu\nu}$?

Compare this to the gravitational force, having three components, but being the derivative of one scalar field. (One d.o.f. giving forces in 3 directions.)

The situation here can be solved similarly, by letting $F^{\mu\nu}$ be the (antisymmetrized) derivative of a four-potential Φ

$$F^{\mu\nu} = \partial^\mu \Phi^\nu - \partial^\nu \Phi^\mu = \Phi^{\nu,\mu} - \Phi^{\mu,\nu}$$

($F_{\mu\nu}$ has to be antisymmetric so we cannot just take $F^{\mu\nu} = \partial^\mu \Phi^\nu$)

From the field equations (*) we then have

$$F^{\mu\nu}{}_{,\mu} = \Phi^{\nu,\mu}{}_{,\mu} - \Phi^{\mu,\nu}{}_{,\mu} = kJ^\nu$$

(Φ^μ is most often called A^μ)

Given boundary conditions on Φ, these four equations are enough to constrain Φ which in turn gives $F^{\mu\nu}$ (Only 4 d.o.f. in Φ)

However, different potentials $\Phi, \tilde{\Phi}$ can give rise to the same fields $F^{\mu\nu}$

Let $\tilde{\Phi} = \Phi - \Psi$ then

$$\begin{aligned} F^{\mu\nu} \to \tilde{F}^{\mu\nu} &= \partial^\mu \tilde{\Phi}^\nu - \partial^\nu \tilde{\Phi}^\mu \\ &= \partial^\mu (\Phi^\nu - \Psi^\nu) - \partial^\nu (\Phi^\mu - \Psi^\mu) \\ &= \partial^\mu \Phi^\nu - \partial^\nu \Phi^\mu - \partial^\mu \Psi^\nu + \partial^\nu \Psi^\mu \\ &= F^{\mu\nu} - \partial^\mu \Psi^\nu + \partial^\nu \Psi^\mu \end{aligned}$$

So if $-\partial^\mu \Psi^\nu + \partial^\nu \Psi^\mu = 0$ we get the same fields $F^{\mu\nu}$

but this is the case if $\Psi^\mu = \partial^\mu \varphi$ for some scalar φ

since then $-\partial^\mu \Psi^\nu + \partial^\nu \Psi^\mu = -\partial^\mu \underbrace{\partial^\nu \varphi + \partial^\nu}_{\text{derivatives commute}} \partial^\mu \varphi = 0$

$\Rightarrow$ We get the same field if Φ is changed by a total derivative $\partial^\mu \varphi$, we have gauge invariance.

In fact we have already seen this in the analytical mechanics part where our gauge invariance condition was

$$\tilde{\Phi}^\mu = \Phi^\mu - \partial^\mu \varphi$$

This ' should not be confused with a Lorentz transformed field

(AM L6:6-7)

$$\begin{cases} V \to V' = V - \dfrac{\partial \Lambda}{\partial t} = V - \dfrac{\partial}{\partial x^0} c\Lambda \\ \bar{A} \to \bar{A}' = \bar{A} + \nabla \Lambda \end{cases}$$

$$\Leftrightarrow \tilde{\Phi}^0 = \Phi^0 - \frac{\partial}{\partial x_0}\varphi \qquad \text{0-component}$$

$$\Leftrightarrow \frac{\tilde{\bar{\Phi}}}{c} = \frac{\bar{\Phi}}{c} + \nabla \frac{\varphi}{c} \qquad \text{spatial components}$$

Divide with c to get Λ : Λ

$$\Phi^0 = V, \quad \varphi = c\Lambda, \quad \bar{\Phi} = c\bar{A}$$

$$\Phi = (V, c\bar{A})$$

There the E and B-fields were given by

$$\bar{E} = -\nabla V - \frac{\partial \bar{A}}{\partial t} \qquad \bar{B} = \nabla \times \bar{A} = \nabla \times \frac{\bar{\Phi}}{c}$$

$$= -\nabla V - \frac{\partial}{\partial \frac{x^0}{c}} \frac{\bar{\Phi}}{c}$$

$$= -\nabla \Phi^0 - \partial_0 \bar{\Phi}$$

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right) = (\partial_1, \partial_2, \partial_3)$$

In component form:

$$E^i = -\partial_i \Phi^0 - \partial_0 \Phi^i = \partial^i \Phi^0 - \partial^0 \Phi^i = F^{i0}$$

$$cB^1 = \partial^3 \Phi^2 - \partial^2 \Phi^3 = F^{32}$$

$$cB^2 = \partial^1 \Phi^3 - \partial^3 \Phi^1 = F^{13}$$

$$cB^3 = \partial^2 \Phi^1 - \partial^1 \Phi^2 = F^{21}$$

$$B^i = \frac{1}{c} \varepsilon^{ijk} \partial_j \Phi^k$$

$$cB^i = -\varepsilon^{ijk} \partial^j \Phi^k$$

$$\begin{cases} \varepsilon^{123} = \varepsilon^{231} = \varepsilon^{312} = 1 \\ \varepsilon^{213} = \varepsilon^{132} = \varepsilon^{321} = -1 \\ \text{all others are } 0 \end{cases}$$

(Since we are equating E and B fields with 3 components to tensors, it is not possible to use a manifestly covariant notation)

We thus see that

$$F^{\mu\nu} = \begin{bmatrix} 0 & F^{01} & F^{02} & F^{03} \\ -F^{01} & 0 & F^{12} & F^{13} \\ -F^{02} & -F^{12} & 0 & F^{23} \\ -F^{03} & -F^{13} & -F^{23} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -cB^3 & cB^2 \\ E^2 & cB^3 & 0 & -cB^1 \\ E^3 & -cB^2 & cB^1 & 0 \end{bmatrix}$$

We conclude that the $\bar{E}$ & $\bar{B}$ fields are part of a rank $\binom{2}{0}$ tensor.

$\Rightarrow$ We know how the $\bar{E}$ & $\bar{B}$ fields transform!

In terms of the $\bar{E}$ & $\bar{B}$ fields, the equation

$$f^\mu = \frac{q}{c} F^{\mu\nu} U_\nu$$

gives

$$f^1 = \frac{q}{c} F^{1\nu} U_\nu = \frac{q}{c}\left(F^{10} U_0 + 0 + F^{12} U_2 + F^{13} U_3 \right)$$

$$= \frac{q}{c}\left(E^1 U^0 - cB^3 U_2 + cB^2 U_3 \right)$$

$U = \gamma(c, \bar{v})$

$$= \frac{q}{\cancel{c}} \gamma \left(E^1 \cancel{c} - \cancel{c}B^3(-v^2) + \cancel{c}B^2(-v^3) \right)$$

From

$$F = \gamma(\bar{v}) \left(\frac{1}{c}\frac{dE}{dt}, \bar{f} \right) \quad \Rightarrow \quad \bar{f} = \frac{1}{\gamma(\bar{v})} \bar{F}$$

we get for the three-force

$$\Rightarrow \bar{f}^1 = \frac{\bar{F}^1}{\gamma} = q\left(E^1 + B^3 v_y - B^2 v_z \right)$$

but this is nothing but the first component of the Lorentz force!

$$(\bar{f})^1 = \left(q\left(\bar{E} + \bar{v} \times \bar{B} \right) \right)^1 = q\left(E_x + v_y B_z - v_z B_y \right)$$

So we have found the correct electric and magnetic fields.

From the field equations $\partial_\mu F^{\mu\nu} = k\rho_0 U^\nu = kJ^\nu$ (*) we get:

(The below two equations determine how E and B are affected by charges.)

- $\nu = 0$

$$\partial_0 F^{00} + \partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30}$$

$$= 0 + \partial_1(E^1) + \partial_2(E^2) + \partial_3(E^3) \overset{(*)}{=} k\rho_0 U^0 \Leftrightarrow \nabla\cdot\bar{E} \overset{U^0=\gamma c}{=} k\rho_0\gamma c = k\rho c$$

$$\nabla\cdot\bar{E} = \frac{\rho}{\varepsilon_0} \Rightarrow k = \frac{1}{\varepsilon_0 c}$$

- $\nu = 1$

$$\partial_0 F^{01} + \partial_1 F^{11} + \partial_2 F^{21} + \partial_3 F^{31} = kJ^1$$

$$-\underbrace{\partial_0}_{\frac{1}{c}\frac{\partial}{\partial t}} E^1 + 0 + \partial_2 cB^3 + \partial_3(-cB^2) = kJ^1$$

$$\nabla\times\bar{B} = \frac{k\bar{J}}{c} + \frac{1}{c^2}\frac{\partial\bar{E}}{\partial t} = \frac{1}{\varepsilon_0 c^2}\bar{J} + \frac{1}{c^2}\frac{\partial\bar{E}}{\partial t}$$

i.e., two of Maxwell's equations, for the others consider:

(The below two equations are consequences of the fact that the field is determined by a four-potential.)

- $\bar{B} = \nabla\times\frac{\bar{\Phi}}{c}$

$$\nabla\cdot\bar{B} = \nabla\cdot\left(\nabla\times\frac{\bar{\Phi}}{c}\right)$$

(Part of Rindler eq. 38.5)

$$= \frac{1}{c}\partial_i\,\varepsilon^{ijk}(\partial_j\Phi^k - \partial_k\Phi^j)$$

$$= \frac{1}{c}\underbrace{\varepsilon^{ijk}}_{\substack{\text{antisym. in } i,j\\ \text{antisym. in } i,k}}(\underbrace{\partial_i\partial_j}_{\text{sym. in } i,j}\Phi^k - \underbrace{\partial_i\partial_k}_{\text{sym. in } i,k}\Phi^j) = 0$$

(There cannot be any magnetic monopoles if electromagnetism is formulated like this)

- $(\nabla\times\bar{E})^1 = \partial_2\bar{E}^3 - \partial_3\bar{E}^2$

(Part of Rindler eq. 38.5)

Recall:

$$F^{\mu\nu} = \partial^\mu\Phi^\nu - \partial^\nu\Phi^\mu$$

$$= \partial_2 F^{30} - \partial_3 F^{20}$$

$$= (\cancel{\partial_2\partial^3\Phi^0} - \partial_2\partial^0\Phi^3) - (\cancel{\underbrace{\partial_3\partial^2}_{\partial^2\partial_3 = --\partial_2\partial^3}\Phi^0} - \partial_3\partial^0\Phi^2)$$

$$= \partial^0(-\underbrace{\partial_2}_{\partial^2}\Phi^3 + \underbrace{\partial_3}_{-\partial^3}\Phi^2) = \partial^0 F^{23} = \frac{1}{c}\frac{\partial}{\partial t}(-cB^1) = -\frac{\partial}{\partial t}B^1$$

Similarly for 2nd and 3rd component $\Rightarrow$ $\nabla\times\bar{E} = -\frac{\partial}{\partial t}\bar{B}$

$\Rightarrow$ We have recovered Maxwell's equations!

Since we can change Φ by a total derivative without changing our $\bar{E}$ & $\bar{B}$ fields, we can choose gauge to make our calculations particularly simple.

One option is to let

$$\partial_\mu \Phi^\mu = 0. \quad \text{Lorentz gauge condition}$$

To see that we can fix this we note that if we choose φ s.t.

$$\partial_\mu \tilde{\Phi}^\mu + \partial_\mu \partial^\mu \varphi = 0$$

then, recalling $\Phi = \tilde{\Phi} + \partial^\mu \varphi$ we get

$$\partial_\mu \Phi^\mu = \partial_\mu \tilde{\Phi}^\mu + \partial_\mu \partial^\mu \varphi = 0 \quad \text{ok}$$

The field equations then give

$$F^{\mu\nu}{}_{,\mu} = \Phi^{\nu,\mu}{}_{,\mu} - \Phi^{\mu,\nu}{}_{,\mu} \quad \left(\overset{(*)}{=} k J^\nu\right)$$

$$= \partial_\mu \partial^\mu \Phi^\nu - \partial_\mu \partial^\nu \Phi^\mu \rightarrow 0$$

$$= \Box \Phi^\nu$$

I.e. we can always choose gauge s.t. $\Rightarrow \Box \Phi^\nu = k J^\nu$

In particular we note that if $J^\nu = 0$

$$\Box \Phi^\nu = 0$$

For $F^{\mu\nu}$ we have

$$\Box F^{\mu\nu} = \Box \left(\partial^\mu \Phi^\nu - \partial^\nu \Phi^\mu\right) = 0$$

$$= \partial^\mu \underbrace{\Box \Phi^\nu}_{0} - \partial^\nu \underbrace{\Box \Phi^\mu}_{0} = 0$$

$$\Rightarrow \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^3}{\partial z^2}\right) \underbrace{F^{\mu\nu}}_{\text{contains E and B-fields}} = 0$$

This is the wave equation, we have recovered that electromagnetic waves travel with the speed of light!