Rindler: 93-96
HUB II.11, 177-179

The free relativistic Lagrangian

Consider a path $\Gamma(\theta)$ through spacetime. From calculus of variations

we know that $\int_\Gamma f(x^\mu, \dot{x}^\mu)\, d\theta$ where $\dot{x}^\mu = \frac{dx^\mu}{d\theta}$

is extremized when $\frac{\partial f}{\partial x^\mu} - \frac{d}{d\theta}\left(\frac{\partial f}{\partial \dot{x}^\mu}\right) = 0$

where we have parametrized the path with some monotonically increasing

parameter θ. Let's now extremize the spacetime distance along a timelike path:

$$\int_\Gamma ds = \int_\Gamma \sqrt{c^2\left(\frac{dt}{d\theta}\right)^2 - \left(\frac{d\bar{x}}{d\theta}\right)^2}\, d\theta = \int_\Gamma \sqrt{(\dot{x}^0)^2 - \dot{\bar{x}}^2}\, d\theta$$

Thus:

in general

$$f = f(x^\mu, \dot{x}^\mu) = \sqrt{(\dot{x}^0)^2 - \dot{\bar{x}}^2}$$

$\mu = 0, 1, 2, 3$

θ, not t, is the underlying integration variable

For the extremizing paths we have from (E)L:

$\mu = 0$, $x^0(\theta), \dot{x}^0(\theta)$

$$\underbrace{\frac{\partial f}{\partial x^0}}_{0} - \frac{d}{d\theta}\overbrace{\left(\frac{1}{2\sqrt{}}\, 2\, \frac{dx^0}{d\theta}\right)}^{\frac{\partial f}{\partial \dot{x}^0}} = 0 \qquad (\dot{x}^0)$$

$\mu = 1, 2, 3$, $x^i(\theta), \dot{x}^i(\theta)$

$$\underbrace{\frac{\partial f}{\partial x^i}}_{0} - \frac{d}{d\theta}\left(\frac{1}{2\sqrt{}}(-2)\frac{dx^i}{d\theta}\right) = 0 \qquad (\dot{\bar{x}})$$

integrate $(\dot{x}^0)$, $(\dot{\bar{x}})$

$$\Rightarrow \quad \frac{\frac{dx^0}{d\theta}}{\sqrt{}} = \alpha, \quad \frac{\frac{d\bar{x}}{d\theta}}{\sqrt{}} = \bar{\beta}$$ for some constants $\alpha, \bar{\beta}$

divide

$$\Rightarrow \quad \frac{\Delta \bar{x}}{\Delta ct} = \frac{\bar{\beta}}{\alpha} \quad \Rightarrow \quad \frac{\Delta \bar{x}}{\Delta t} = \text{const}$$

So straight lines through spacetime extermize the Lorentz invariant distance!

It seems reasonable that free particles should follow straight lines, and it seems reasonable that the thing to extremize is the Lorentz invariant spacetime distance.

Let us therefore assume that free particles extremize

$$\int_\Gamma ds = \int \underbrace{\sqrt{(\dot{x}^0)^2 - \dot{\vec{x}}^2}}_{f}\, d\theta$$

for any monotonically increasing θ

Note that since f doesn't depend on x^μ i.e., we have translation invariance in space and time, we have conserved quantities, namely:

$$\frac{d}{d\theta}\left(\frac{\partial f}{\partial \dot{x}^\mu}\right) = 0 \quad \Rightarrow \frac{\partial f}{\partial \dot{x}^\mu}$$ is conserved.

Remark: Now space and time enter on equal footing!

This is an example of Noether's theorem which states that whenever the Lagrangian is invariant under a continuous transformation, we have a conserved quantity.

We will find that the Lagrangian for free particles is proportional to f.

Let's calculate the conserved quantities, from x^0, $\bar{x}$ we have:

$$\frac{\partial f}{\partial \dot{x}^\mu} = \frac{1}{\sqrt{(\dot{x}^0)^2 - \dot{\bar{x}}^2}} \frac{dx_\mu}{d\theta}$$

(Eq. $\bar{x}$ has an extra -, absorbed by writing as dx_μ .)

Taking $\theta = \tau$ gives

$$\frac{\partial f}{\partial \dot{x}^\mu} = \frac{1}{\sqrt{\left(\frac{dx^0}{d\tau}\right)^2 - \left(\frac{d\bar{x}}{d\tau}\right)^2}} \frac{dx_\mu}{d\tau}$$

def of $u^\mu \rightarrow$

$$= \frac{1}{\sqrt{\underbrace{u^{0\,2} - \bar{u}^2}_{c^2}}} \frac{dx_\mu}{d\tau}$$

$$= \frac{1}{c} u_\mu$$

$$\Rightarrow \frac{d}{d\tau} \frac{\partial f}{\partial \dot{x}^\mu} = \frac{d}{d\tau}\left[\frac{1}{c} u_\mu\right]$$

$$= \frac{1}{c} \frac{du_\mu}{d\tau}$$

Multiplying with mc, and raising μ we get

$$0 = m_0 \frac{du^\mu}{d\tau} = \frac{dP^\mu}{d\tau}$$

But this is conservation of 4-momentum!

We thus find that 4-momentum conservation for free particles follows from assuming that particles follow paths that extremize the invariant spacetime distance, and noting that this distance is translation invariant.

In particular, now we get energy conservation as well!

We may guess that the free action is $-mc\int_\Gamma ds$

giving

(the minus sign will soon be explained)

(typos in eq. II.11.11 in HUB)

$$\int_\Gamma L_{free}\, dt = -mc\int_\Gamma ds$$

$$= -mc^2\int_\Gamma d\tau$$

$$= \int -mc^2\left(1-\frac{\bar{v}^2}{c^2}\right)^{1/2} dt$$

$$\Rightarrow \boxed{L_{free} = -mc^2\sqrt{1-\frac{\bar{v}^2}{c^2}} = \frac{-mc^2}{\gamma}}$$

For small velocities we may expand the result:

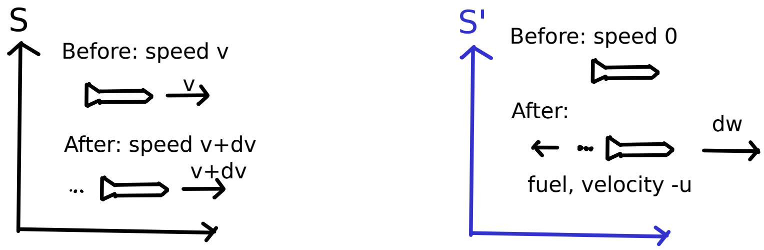
$$L_{free} = -mc^2\sqrt{1-\frac{\bar{v}^2}{c^2}} = -mc^2 + \frac{1}{2}m\bar{v}^2 + \cdots$$

We thus recover the free non-relativistic Lagrangian minus a constant mass term.

Ex: Consider a relativistic rocket with speed v which accelerates by ejecting relativistic fuel.

(HUB II.10 172-175)

Consider the acceleration infinitesimally



In its momentary rest frame S' (i.e. rest frame before emission of fuel) the rocket starts at rest and has after the emission of fuel speed

$$dw \quad (\neq dv)$$

In S' we have

- rocket before $\quad \underline{p} = (Mc, 0, 0, 0)$
- rocket after $\quad \underline{p}' = \underbrace{\gamma(dw)}_{\approx 1 \text{ as dw infinitesimal}} (M + dM)c\,(1, 0, 0, \frac{dw}{c})$ $\quad$ I take dM to be negative
- fuel after $\quad \underline{p}'' = \underbrace{\gamma(-u)}_{\text{velocity of fuel}} \underbrace{dm}_{\text{mass of fuel}} c\,(1, 0, 0, -\frac{u}{c})$

Conservation of 4-momentum gives

$$\underline{p}'' = \underline{p} - \underline{p}'$$

Component 0: $\quad \gamma(-u)\, dm\, c = \cancel{Mc} - (\cancel{M} + dM)c = -dMc$

Component 3: $\quad \gamma(-u) dm\, c \left(\frac{-u}{c}\right) = -(M + dM)\cancel{c}\frac{dw}{\cancel{c}}$

$$\Rightarrow dw = +\frac{dM\cancel{c}}{\underbrace{M + dM}_{\approx M,\ dM \text{ infinitesimal}}}\left(\frac{-u}{\cancel{c}}\right) \approx -u\frac{dM}{M}$$

In S we get the speed by velocity addition:

$$\cancel{v} + dv = \frac{v + dw}{1 + \frac{v\,dw}{c^2}} \qquad \text{Standard form for parallel velocity addition}$$

dw infinitesimal $\quad \approx (v + dw)\left(1 - \frac{v\,dw}{c^2}\right)$

ignore $(dw)^2$ $\quad \approx \cancel{v} + dw\left(1 - \frac{v^2}{c^2}\right)$

Solving for dv we get:

$$\Rightarrow \quad dv = dw\left(1-\frac{v^2}{c^2}\right) \overset{dw = -u\frac{dM}{M}}{=} -u\,\frac{dM}{M}\left(1-\frac{v^2}{c^2}\right)$$

$$\Rightarrow \int_{v_i}^{v_f} \underbrace{\frac{dv}{1-\frac{v^2}{c^2}}}_{>0} = -\int_{M_i}^{M_f} u\,\frac{dM}{M} = u\log\frac{M_i}{M_f} \qquad M_i > M_f$$

$v_f > v_i$

$$\frac{1}{1-\frac{v^2}{c^2}} = \frac{1}{2}\,\frac{\left(1+\frac{v}{c}\right)+\left(1-\frac{v}{c}\right)}{\left(1+\frac{v}{c}\right)\left(1-\frac{v}{c}\right)} = \frac{1}{2}\,2\,\frac{1}{1-\frac{v^2}{c^2}}$$

$$\| $$

$$\int_{v_i=0}^{v_f} \frac{1}{2}\left(\frac{1}{1-\frac{v}{c}}+\frac{1}{1+\frac{v}{c}}\right)dv =$$

Let $v_i = 0$

$$= \frac{c}{2}\left[-\log\left(1-\frac{v}{c}\right)\Big|_0^{v_f} + \log\left(1+\frac{v}{c}\right)\Big|_0^{v_f}\right]$$

compensate for inner derivatives

$$= \frac{c}{2}\left[-\log\left(1-\frac{v_f}{c}\right)+\log\left(1+\frac{v_f}{c}\right)\right] = \frac{c}{2}\log\left(\frac{1+\frac{v_f}{c}}{1-\frac{v_f}{c}}\right)$$

$$e^{LHS} = e^{RHS} \Rightarrow \left(\frac{M_i}{M_f}\right)^u = \left(\frac{1+\frac{v_f}{c}}{1-\frac{v_f}{c}}\right)^{\frac{c}{2}}$$

raise to power 2/c and multiply with $1-\frac{v_f}{c}$

$$\Rightarrow \left(\frac{M_i}{M_f}\right)^{u\frac{2}{c}}\left(1-\frac{v_f}{c}\right) = 1+\frac{v_f}{c}$$

$$\Rightarrow \frac{v_f}{c}\left[1+\left(\frac{M_i}{M_f}\right)^{\frac{2u}{c}}\right] = \left(\frac{M_i}{M_f}\right)^{\frac{2u}{c}} - 1$$

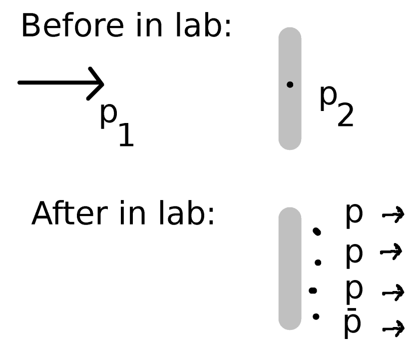
$$\Rightarrow v_f = \frac{\left(\frac{M_i}{M_f}\right)^{\frac{2u}{c}} - 1}{\underbrace{\left(\frac{M_i}{M_f}\right)}_{>1}^{\frac{2u}{c}} + 1}\cdot c$$

- Speed of fuel u=c gives maximal speed.
- We also note that $\frac{v_f}{c} \xrightarrow[\frac{M_i}{M_f}\to\infty]{} 1$

HUB p. 162-164
(II.10)

Ex: the proton anti-proton creation from lecture 4:

$p+p \to p+p+\bar{p}+p$.



In lab:

$$\begin{aligned} P &= P_1 + P_2 \\ &= \gamma(m_p c, 0, 0, m_p v) + (m_p c, 0, 0, 0) \\ &= m_p c\left(\gamma+1, 0, 0, \gamma \tfrac{v}{c}\right) \end{aligned}$$

Last time we used the invariant mass square to calculate the threshold energy, alternatively we can boost to the CM-system:

$$\bar{\beta} = \frac{\bar{P}_{lab}\, c}{E_{lab}} = \frac{\cancel{m_p c}\left(0,0,\gamma\frac{v}{c}\right)\cancel{c}}{\cancel{m_p c^2}(\gamma+1)} = \frac{\gamma}{\gamma+1}\left(0,0,\frac{v}{c}\right)$$

But then the gamma factor for boosting to the CM-system is:

$$\gamma_\beta = \frac{1}{\sqrt{1-\bar{\beta}^2}} = \frac{1}{\sqrt{1-\frac{\gamma^2}{(\gamma+1)^2}\frac{v^2}{c^2}}} = \frac{\gamma+1}{\sqrt{(\gamma+1)^2-\gamma^2\frac{v^2}{c^2}}} = \frac{\gamma+1}{\sqrt{\underbrace{\gamma^2\left(1-\frac{v^2}{c^2}\right)}_{1}+2\gamma+1}} = \frac{\sqrt{\gamma+1}}{\sqrt{2}}$$

Note!
different γ

In the CM-system we know: $E_{CM} = 4 m_p c^2$, $\bar{P}_{CM} = 0$

and that half the energy must come from the target proton,

$$E_{CM,2} = \gamma_\beta m_p c^2 \overset{!}{=} 2 m_p c^2 \Rightarrow \gamma_\beta = 2$$

$$\Leftrightarrow 2 = \frac{\sqrt{\gamma+1}}{\sqrt{2}} \qquad \Rightarrow 2^2 2 = (\gamma+1) \Rightarrow \gamma = 7$$

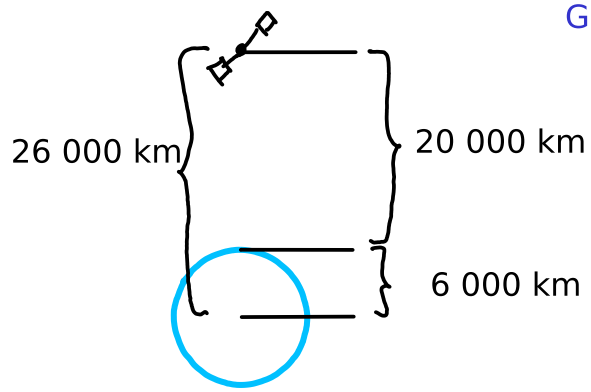
We find the same gamma factor as last time:

$$\gamma = 7 \Rightarrow E_{p_1} = \gamma m_p c^2 = 7 m_p c^2$$

Collisions with different particle content on the incoming and outgoing side are called inelastic, whereas collisions with the same particles are called elastic.

Ex: A GPS satellite 20 000 km above ground orbits the earth in 12 hours.

Estimate the error in position arising during one day due to special relativistic effects, as the clock of the satellite appears slow on the Earth.



Relative to an observer on the surface of the earth the speed of the satellite will vary, but let us - for an estimate - assume

$$v_s = \frac{2\pi \, 26000 \text{ km}}{12 \cdot 60^2 \text{ s}} \approx 3{,}8 \text{ km/s}$$

The error in time is then

$$\Delta t_e - \Delta t_s = 24\text{h} - 24\text{h}\sqrt{1-\frac{v^2}{c^2}} \approx 24\text{h}\left(\cancel{1} - \cancel{1} + \frac{1}{2}\frac{v^2}{c^2}\right)$$

$$\approx 7 \, \mu\text{s}$$

Which in position corresponds to $7\,\mu\text{s} \cdot 3 \cdot 10^8 \frac{\text{m}}{\text{s}} \approx 2 \text{ km} \,!$

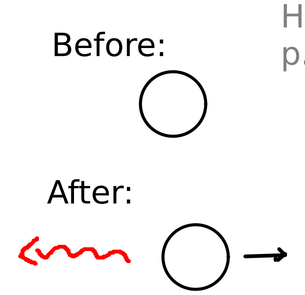
Conclusion: For reasonable accuracy special relativistic effects must be taken into account.

Remark: The error in time due to general relativity is much larger

$\sim 40 \, \mu\text{s}.$

Ex: Consider the energy of a photon emitted from an atom at rest

We have the 4-momenta:



Before, atom: $\underline{P} = Mc\,(1, 0, 0, 0)$

After, atom: $\underline{P}' = \gamma \overbrace{(M - \Delta M)}^{\text{new mass}} c\,(1, 0, 0, \beta)$

After, photon: $\underline{P}_\gamma = \frac{E_\gamma}{c}\,(1, 0, 0, 1)$

4-momentum conservation =>

$$\underline{P} = \underline{P}' + \underline{P}_\gamma$$

$$\Rightarrow \underline{P}'^2 = (\underline{P} - \underline{P}_\gamma)^2$$

With p' alone squaring is easier

$$\Rightarrow (M - \Delta M)^2 c^2 = \underbrace{\underline{P}^2}_{M^2c^2} + \underbrace{\underline{P}_\gamma^2}_{0} - 2\underbrace{\underline{P} \cdot \underline{P}_\gamma}_{P^0 P_{\gamma 0}}$$

as $\bar{p} = 0$

$$= M^2 c^2 - 2 M \cancel{c} \frac{E_\gamma}{\cancel{c}}$$

$$\Rightarrow E_\gamma = \frac{\left(M^2 - (M - \Delta M)^2\right) c^2}{2M}$$

$$= \frac{\left(2 M \Delta M - (\Delta M)^2\right) c^2}{2M}$$

$$= \left(1 - \frac{\Delta M}{2M}\right) \Delta M c^2$$

When the atom emits the photon, the electron jumps from an orbit m

with higher energy to an orbit n with lower energy.

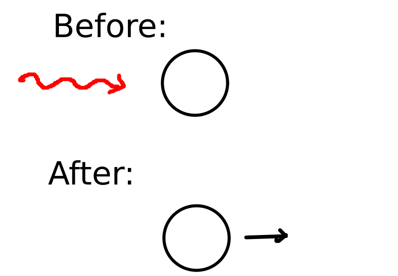
The difference is: $\Delta M c^2 = E_n - E_m = E_{nm}$

but $E_\gamma = E_{nm} \left(1 - \frac{\Delta M}{2M}\right) = E_{nm} \left(1 - \frac{E_{nm}}{2Mc^2}\right)$

The photon thus gets somewhat lower energy than E_{nm}

Due to the recoil of the atom it "looses" $\frac{E_{nm}^2}{2Mc^2}$

Ex: An atom at rest absorbs a photon, compare the energy of the absorbed photon to what we just have seen.



Before, atom: $\quad p = Mc(1, 0, 0, 0)$

Before photon: $\quad p_\gamma = \frac{E_\gamma}{c}(1, 0, 0, 1)$

After atom: $\quad p' = \gamma\underbrace{(M + \Delta M)}c(1, 0, 0, \beta)$

The energy of the atom,
its mass will increase slightly

Four-momentum conservation gives:

$$p' = p_\gamma + p$$

$$\Rightarrow p'^2 = (p_\gamma + p)^2$$

$$(M+\Delta M)^2 c^2 = \underbrace{p^2}_{M^2c^2} + 2 p_\gamma \cdot p + \underbrace{p_\gamma^2}_{0}$$

$$= M^2c^2 + 2p^0 p_{\gamma 0} + \underbrace{p^i p_{\gamma i}}_{0} \quad \text{since } p^i = 0$$

$$= M^2c^2 + 2M\cancel{c}\frac{E_\gamma}{\cancel{c}}$$

$$\Rightarrow E_\gamma = \frac{(M+\Delta M)^2c^2 - M^2c^2}{2M} = \frac{(2M\Delta M + (\Delta M)^2)c^2}{2M}$$

$$= \left(1 + \frac{\Delta M}{2M}\right)\Delta M c^2$$

i.e. the photon needs more energy than E_{nm}

Question: Does this imply that an atom gas is transparent to its own radiation?

No, since

1) The emission spectrum has a certain width.

2) At normal temperatures the atom's kinetic energy from motion is much larger than the difference, $E_{\substack{kin\\atom}} > E_{nm}\frac{\Delta M}{2M}$