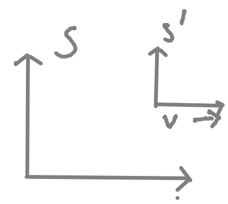
Special Relativity, repetition

Assuming homogeneity and isotropy and that the speed of light is the same for all observers we derived the Lorentz transformation for a boost in x-direction:

$$t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

$$x' = \gamma(x - vt)$$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$



The transverse components don't change y'=y, z'=z.

One can see that this leads to Lorentz contraction, i.e. lenghts in moving systems seem to be contracted in the direction of motion

$$\underbrace{l_{\parallel}}_{\text{length in other systems seems contracted}} = \sqrt{1-\frac{v^2}{c^2}}\;\underbrace{l'_{\parallel}}_{\text{rest length}}, \qquad l_{\perp} = l'_{\perp}$$

Warning:
The Lorentz-contraction and time dilation equations are dangerous as one needs to understand which system is the moving system.
Each observer thinks that others rods are short, and that others clocks are slow.

To agree on the speed of light (as lengths are shortened) observers must also disagree on time. Time seems to go slower for moving observers

$$\underbrace{\Delta t}_{\text{observer}} = \underbrace{\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}}_{>1}\;\underbrace{\Delta t'}_{\text{moving}}$$

this time seems to changes slower

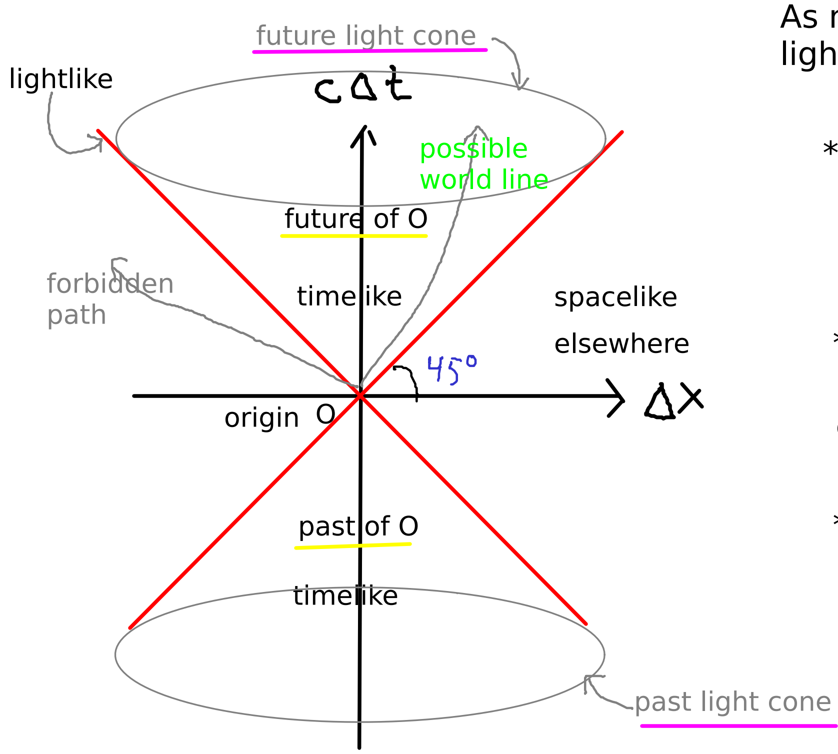
time dilation

We also note that Lorentz transformations don't change the invariant spacetime distance:

$$(\Delta s)^2 = c^2(\Delta t)^2 - (\Delta \vec{x})^2$$

All observers thus agree on the value of this.

As nothing travels faster than the speed of light we may sort spacetime distances into



* spacelike $c^2(\Delta t)^2 - (\Delta \vec{x})^2 < 0$

Events with spacelike separation are causally disconnected

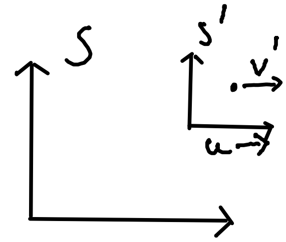
* timelike $c^2(\Delta t)^2 - (\Delta \vec{x})^2 > 0$

Events with timelike separation are causally connected

* lightlike $c^2(\Delta t)^2 - (\Delta \vec{x})^2 = 0$

From the Lorentz transformation we may derive the relativistic velocity addition formula

$$V = \frac{u + v'}{1 + \frac{uv'}{c^2}}$$



For photons the wavelegth of the observer relates to the wavelength of the emitter as:

$$\lambda_{obs} = \lambda_{em}\sqrt{\frac{1+\frac{u}{c}}{1-\frac{u}{c}}}$$

where u is counted positive if the observer is moving away from the emitter.

Introducing $x^0 = ct$ and $\beta = \frac{v}{c}$ we may write the x-direction boost as:

$$\begin{bmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{bmatrix} = \underbrace{\begin{bmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{bmatrix}}_{\Lambda} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \qquad x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

This means that the vector

$$x^\mu = (x^0, x^1, x^2, x^3)$$

transforms as (i.e. is) a contravariant 4-vector. The vector

$$(x_0, x_1, x_2, x_3) = (x^0, -x^1, -x^2, -x^3) = x_\mu$$

is a covariant 4-vector.

The contravariant and covariant 4-vectors are related via the metric

$$x_\mu = g_{\mu\nu} x^\nu, \qquad x^\mu = g^{\mu\nu} x_\nu$$

where we use the Einstein summation convention; we sum over repeated indices.

The (covariant) metric $g_{\mu\nu}$ as well as its inverse, the contravariant metric $g^{\mu\nu}$

may in special relativity be written as

$$g = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix} = g^{-1}$$

(not so in general relativity)

All contravariant vectors transform in the same way, meaning, in particular, that for the four-momentum

$$p^\mu = m u^\mu = m \frac{dx^\mu}{d\tau} = (\underbrace{p^0}_{E/c}, p^1, p^2, p^3) = m\gamma(c, \bar{v}),$$

where $d\tau = \sqrt{(dt)^2 - \frac{(d\bar{x})^2}{c^2}}$ is the proper time (i.e. time measured by own clock),

we have:

$$\begin{array}{c} \\ E'/c \\ \\ \\ \\ \end{array} \begin{bmatrix} p'^0 \\ p'^1 \\ p'^2 \\ p'^3 \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & \text{same as above} & & \\ & & & 1 \end{bmatrix} \begin{bmatrix} p^0 \\ p^1 \\ p^2 \\ p^3 \end{bmatrix} \begin{array}{l} \leftarrow E/c \\ \leftarrow p^x \\ \leftarrow p^y \\ \leftarrow p^z \end{array}$$

We thus know how energy and momentum in different Lorentz frames are related.

The spatial components of the 4-momentum form the relativistic 3-momentum:

$$\bar{p} = m\gamma\bar{v}$$

and the 0-component is related to the energy

$$E = p^0 c = m\gamma c^2$$

We know that the four-momentum square must be a Lorentz scalar, giving

$$p \cdot p = p^\mu p_\mu = (p^0)^2 - (\bar{p})^2$$

$$= \underbrace{(\gamma m c)^2}_{E^2/c^2} - \underbrace{(\gamma m \bar{v})^2}_{\bar{p}^2} = m^2\gamma^2\underbrace{\left(1 - \frac{\bar{v}^2}{c^2}\right)}_{1} c^2 = m^2c^2$$

or $$E^2 = \bar{p}^2c^2 + m^2c^4$$

As $p^\mu = m u^\mu$ we note that the 4-velocity u^μ square is c^2

It is always possible to boost to the center of momentum system using

$$\bar{\beta} = \frac{\bar{P}_{tot}c}{E_{tot}}$$

In general we have that the total 4-momentum is conserved

$$P^\mu = \sum_{i,\ \text{incoming}} p_i^\mu = \sum_{o,\ \text{outgoing}} p_o^\mu$$

A useful quantity for calculations with particles is the invariant mass square

$$P^\mu P_\mu = c^2 M^2$$

This quantity is both conserved and a Lorentz scalar.

Scalars, the co- and contravariant 4-vectors, the co- and contravariant metric are special cases of tensors. A tensor may have m indices transforming contravariantly and n indices transforming covariantly.

A general rank $\binom{m}{n}$ tensor transform as

$$T'^{\overbrace{\mu\cdots\nu}^{\text{m indices}}}{}_{\underbrace{\alpha\cdots\beta}_{\text{n indices}}} = T^{\tilde{\mu}\cdots\tilde{\nu}}{}_{\tilde{\alpha}\cdots\tilde{\beta}} \left(\frac{\partial x'^\mu}{\partial x^{\tilde{\mu}}} \cdots \frac{\partial x'^\nu}{\partial x^{\tilde{\nu}}}\right)\left(\frac{\partial x^{\tilde{\alpha}}}{\partial x'^\alpha} \cdots \frac{\partial x^{\tilde{\beta}}}{\partial x'^\beta}\right)$$

In general proper Lorentz transformations can be generated using 6 parameters

$$\Lambda = e^{L} \qquad L = +\bar{\omega}\cdot\bar{J} - \bar{\eta}\cdot\bar{K}$$

$$\bar{\omega} = (\omega_1, \omega_2, \omega_3), \quad \bar{\eta} = (\eta_1, \eta_2, \eta_3)$$

$$J_{\underset{23}{1}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad J_{\underset{13}{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad J_{\underset{12}{3}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K_{\underset{01}{1}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad K_{\underset{02}{2}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad K_{\underset{03}{3}} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

The matrices $J_1 J_2 \ J_3$ generate rotationsand $K_1 \ K_2 K_3$ generate boosts.

The Lorentz boost can also be written in terms of the rapidity

$$\eta = \operatorname{arctanh}\left(\frac{v}{c}\right)$$

$$\begin{pmatrix} x'^0 \\ x'^1 \end{pmatrix} = \begin{pmatrix} \cosh(\eta) & -\sinh(\eta) \\ -\sinh(\eta) & \cosh(\eta) \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix}$$

Rapidities for parallel boosts are additive

$$\eta_{tot} = \eta_1 + \eta_2$$

The relativistic Lagrangian for a free particle is:

$$L_{free} = -mc^2\sqrt{1 - \frac{v^2}{c^2}} = -\frac{mc^2}{\gamma}$$