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Orthogonal multiplet bases in color space

In collaboration with
Stefan Keppeler (Tübingen),
JHEP09(2012)124, arXiv:1207.0609

- Motivation
- The standard method (trace basis)
- Orthogonal multiplet bases for $SU(N_c)$
- Conclusion and outlook

Tübingen
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Malin Sjödaahl

Motivation

- With the start of the LHC follows an increased demand of accurately calculated processes in QCD
- This is applicable to [NLO calculations](#) and [resummation](#)
- ...but my perspective is from a [parton shower](#) point of view
- First SU(3) parton shower in collaboration with [Simon Plätzer](#)
JHEP 07(2012)042, color structure treated using my C++
[ColorFull](#) code



The color space

- We never observe individual colors \rightarrow we are only interested in color summed quantities
- For given external partons, the color space is a finite dimensional **vector space** equipped with a scalar product

$$\langle A, B \rangle = \sum_{a,b,c,\dots} (A_{a,b,c,\dots})^* B_{a,b,c,\dots}$$

Example: If

$$A = \sum_g (t^g)^a{}_b (t^g)^c{}_d = \sum_g {}^a_b \text{---} g \text{---} {}^c_d ,$$

$$\text{then } \langle A|A \rangle = \sum_{a,b,c,d,g,h} (t^g)^b{}_a (t^g)^d{}_c (t^h)^a{}_b (t^h)^c{}_d$$

- We may use any basis (spanning set)



The standard treatment

- Every 4g vertex can be replaced by 3g vertices:

$$\begin{array}{c} a, \alpha \\ \diagup \\ \diagdown \\ c, \gamma \end{array} \begin{array}{c} b, \beta \\ \diagdown \\ \diagup \\ d, \delta \end{array} = \begin{array}{c} \diagup \\ \diagdown \\ \diagup \\ \diagdown \end{array} + \begin{array}{c} \diagup \\ \diagdown \\ \diagup \\ \diagdown \end{array} + \begin{array}{c} \diagup \\ \diagdown \\ \diagup \\ \diagdown \end{array}$$

$$\times ig_s^2 (g^{\alpha\delta} g^{\beta\gamma} - g^{\alpha\gamma} g^{\beta\delta}) \quad \times ig_s^2 (g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\delta} g^{\beta\gamma}) \quad \times ig_s^2 (g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\gamma} g^{\beta\delta})$$

(read counter clockwise)

- Every 3g vertex can be replaced using:

$$\begin{array}{c} a \\ \diagup \\ \diagdown \\ b \quad c \end{array} = \frac{1}{T_R} \left(\begin{array}{c} \diagup \\ \diagdown \\ \diagup \\ \diagdown \end{array} - \begin{array}{c} \diagup \\ \diagdown \\ \diagup \\ \diagdown \end{array} \right)$$

if_{abc}

- After this every internal gluon can be removed using:

$$\begin{array}{c} \diagup \\ \diagdown \end{array} = T_R \begin{array}{c} \diagup \\ \diagdown \end{array} - \frac{T_R}{N_c} \begin{array}{c} \diagup \\ \diagdown \end{array}$$



- This [can be applied to any QCD amplitude](#), tree level or beyond
- For gluons at tree level, the result is a sum over closed quark-lines

$$A = \sum_{\sigma \in S_{N_g-1}} A_\sigma \text{ (diagram with } N_g \text{ gluon lines and a closed quark loop)} = \sum_{\sigma \in S_{N_g-1}} A_\sigma \text{Tr}[t^1 t^{\sigma(2)} \dots t^{\sigma(N_g)}]$$

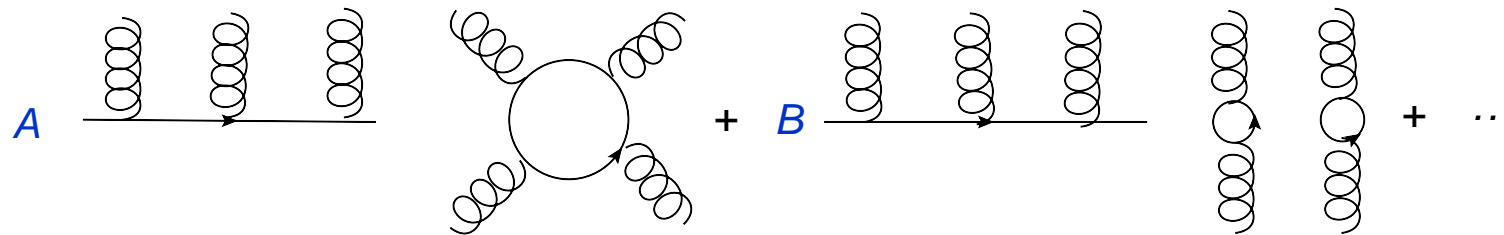
- At one loop we may have a product of up to two traces, and for arbitrary order up to $N_g/2$ traces
- For processes with quarks there are open quark lines as well:
For example for 2 (incoming + outgoing) gluons and one $q\bar{q}$ pair

$$\text{(diagram of a blob with two gluon lines)} = A_1 \text{ (diagram 1)} + A_2 \text{ (diagram 2)} + A_3 \text{ (diagram 3)}$$

(an incoming quark is the same as an outgoing anti-quark)



- In general an amplitude can be written as linear combination of different color structures, like

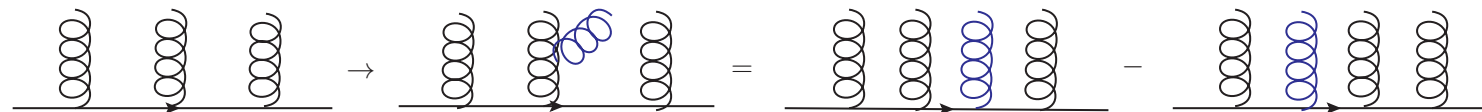


- This is the kind of “trace bases” used in [ColorFull](#) for the SU(3) parton shower, and in most NLO calculations



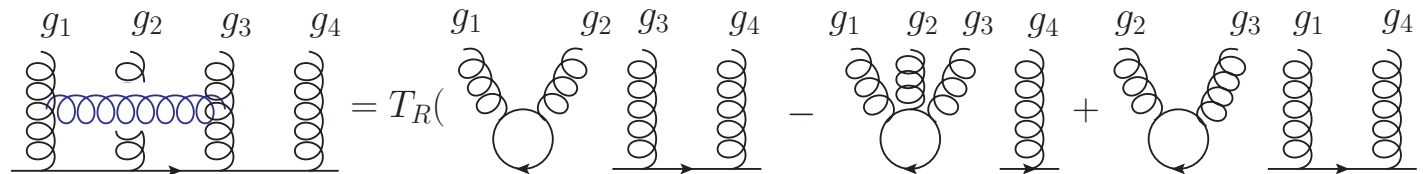
It has some nice properties

- The effect of gluon emission is easily described:



Convention: + when inserting after, minus when inserting before.

- So is the effect of gluon exchange:



Convention: + when inserting after, - when inserting before



ColorFull

For the purpose of treating a general QCD color structure I have written a C++ color algebra code, [ColorFull](#), which:

- Is used in the color shower with Simon Plätzer
- automatically creates a “trace basis” for any number and kind of partons, and to any order in α_s
- describes the effect of gluon emission
- ... and gluon exchange
- squares color amplitudes
- can be used with boost for optimized calculations
- is planned to be published separately



However...

- This type of “basis” is **non-orthogonal** and **overcomplete**
(for more than N_c gluons plus $q\bar{q}$ -pairs)
- ... and the number of basis vectors grows as a factorial in $N_g + N_{q\bar{q}}$
→ when squaring amplitudes we run into a factorial square scaling
- Hard to go beyond ~ 8 gluons plus $q\bar{q}$ -pairs



However...

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→ when squaring amplitudes we run into a factorial square scaling
- Hard to go beyond ~ 8 gluons plus $q\bar{q}$ -pairs
- **Would be nice with minimal orthogonal basis**



Orthogonal multiplet bases

In collaboration with Stefan Keppeler

- The color space may be decomposed into irreducible representations, enumerated using Young tableaux multiplication
- For example for $qq \rightarrow qq$ we have

$$\begin{array}{c} \square \\ 3 \end{array} \otimes \begin{array}{c} \square \\ 3 \end{array} = \begin{array}{cc} \square & \square \\ 6 \end{array} \oplus \begin{array}{c} \square \\ \bar{3} \end{array}$$

and the corresponding basis vectors

$$\begin{array}{c} \rightarrow \\ \rightarrow \\ \rightarrow \end{array} \begin{array}{|c|} \hline \\ \hline \end{array} \begin{array}{c} \rightarrow \\ \rightarrow \\ \rightarrow \end{array} = \frac{1}{2} \begin{array}{c} \rightarrow \\ \rightarrow \\ \rightarrow \end{array} \begin{array}{c} \rightarrow \\ \rightarrow \\ \rightarrow \end{array} + \frac{1}{2} \begin{array}{c} \rightarrow \\ \rightarrow \\ \rightarrow \end{array} \begin{array}{c} \rightarrow \\ \rightarrow \\ \rightarrow \end{array}, \quad \begin{array}{c} \rightarrow \\ \rightarrow \\ \rightarrow \end{array} \begin{array}{|c|} \hline \\ \hline \end{array} \begin{array}{c} \rightarrow \\ \rightarrow \\ \rightarrow \end{array} = \frac{1}{2} \begin{array}{c} \rightarrow \\ \rightarrow \\ \rightarrow \end{array} \begin{array}{c} \rightarrow \\ \rightarrow \\ \rightarrow \end{array} - \frac{1}{2} \begin{array}{c} \rightarrow \\ \rightarrow \\ \rightarrow \end{array} \begin{array}{c} \rightarrow \\ \rightarrow \\ \rightarrow \end{array}$$

Here Cvitanović's [birdtrack notation](#) is used. These color tensors are orthogonal both when seen as qq -projectors, and when seen as basis vectors on the 4-parton space



- For quarks we can construct orthogonal projectors and basis vectors using Young tableaux ...at least from the Hermitian quark projectors
- In fact the $qq \rightarrow qq$ color space is the same as for $q\bar{q} \rightarrow q\bar{q}$,

$$\square \otimes \bar{\square} = \bullet \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}$$

and we could as well have used the basis:

$$\mathbf{V}^1 = \delta^a_b \delta^c_d = \begin{array}{c} a \\ \curvearrowright \\ b \end{array} \quad \begin{array}{c} \curvearrowleft \\ c \\ d \end{array}, \quad \mathbf{V}^8 = (t^g)^a_b (t^g)^c_d = \begin{array}{c} a \\ \curvearrowright \\ b \end{array} \quad \begin{array}{c} \text{gluon} \\ c \\ d \end{array}$$

- In general we may “comb” the involved particles as incoming and outgoing as we wish
- In QCD we have quarks, anti-quarks and gluons
→ No obvious way to construct projectors



The simplest gluon example, $gg \rightarrow gg$

- Basis vectors can be enumerated using Young tableaux multiplication

$$\begin{array}{c} \square \square \\ \square \end{array} \otimes \begin{array}{c} \square \square \\ \square \end{array} = \begin{array}{c} \bullet \\ 1 \end{array} \oplus \begin{array}{c} \square \square \\ \square \end{array} \oplus \begin{array}{c} \square \square \\ \square \end{array} \oplus \begin{array}{c} \square \square \square \\ 10 \end{array} \oplus \begin{array}{c} \square \square \square \\ \square \end{array} \oplus \begin{array}{c} \square \square \square \square \\ 27 \end{array} \oplus \begin{array}{c} 0 \\ 0 \end{array}$$

- As color is conserved an incoming multiplet of a certain kind can only go to an outgoing multiplet of the same kind,
 $1 \rightarrow 1, 8 \rightarrow 8 \dots$

Charge conjugation implies that some vectors only occur together...



The problem is the construction of the corresponding projection operators; the Young tableaux operate with “quark-units”

- Problem first solved for two gluons by MacFarlane, Sudbery, and Weisz 1968, however only for $N_c = 3$
- General N_c solution for two gluons by Butera, Cicuta and Enriotti 1979
- General N_c solution for two gluons by Cvitanović, in group theory books, 1984 and 2008, using polynomial equations
- General N_c solution for two gluons by Dokshitzer and Marchesini 2006, using symmetries and intelligent guesswork



$$\mathbf{P}^1 = \frac{1}{N_c^2 - 1} \text{ (gluon loop) }, \mathbf{P}^{8s} = \frac{N_c}{2T_R(N_c^2 - 4)} \text{ (gluon exchange) }, \mathbf{P}^{8a} = \frac{1}{2N_c T_R} \text{ (gluon exchange with vertices) },$$

$$\mathbf{P}^{10} = \frac{1}{2} \text{ (gluon box) } + \frac{1}{2T_R^2} \text{ (gluon box with loop) } - \frac{1}{2} \mathbf{P}^{8a}$$

$$\mathbf{P}^{\overline{10}} = \frac{1}{2} \text{ (gluon box) } - \frac{1}{2T_R^2} \text{ (gluon box with loop) } - \frac{1}{2} \mathbf{P}^{8a}$$

$$\mathbf{P}^{27} = \frac{1}{2} \text{ (gluon box) } + \frac{1}{2T_R^2} \text{ (gluon box with loop) } - \frac{N_c - 2}{2N_c} \mathbf{P}^{8s} - \frac{N_c - 1}{2N_c} \mathbf{P}^1$$

$$\mathbf{P}^0 = \frac{1}{2} \text{ (gluon box) } - \frac{1}{2T_R^2} \text{ (gluon box with loop) } - \frac{N_c + 2}{2N_c} \mathbf{P}^{8s} - \frac{N_c + 1}{2N_c} \mathbf{P}^1$$



New idea: Could this work in general?

On the one hand side

$$g_1 \otimes g_2 \otimes \dots \otimes g_n \subseteq (q_1 \otimes \bar{q}_1) \otimes (q_2 \otimes \bar{q}_2) \otimes \dots \otimes (q_n \otimes \bar{q}_n)$$

so there is hope...

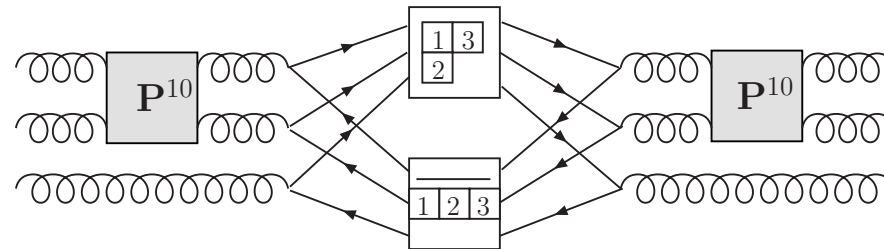
On the other hand...

- Why should it?
- In general there are many instances of a multiplet, how do we know we construct all?
- Even if such a decomposition would give the new multiplets (which could not be present for fewer gluons) in a unique way, we would have to project out all instances of all “old” multiplets. How do we get them?



Key observation:

- Starting in a given multiplet, corresponding to some $q\bar{q}$ symmetries, such as 10, from $\boxed{1\ 2} \otimes \overline{\boxed{1\ 2}}$, it turns out that **for each way of attaching a quark box to $\boxed{1\ 2}$ and an anti-quark box to $\overline{\boxed{1\ 2}}$, to there is at most one new multiplet!** For example, the projector $\mathbf{P}^{10,35}$ can be seen as coming from



after having projected out "old" multiplets

- In fact, **for large enough N_c , there is precisely one new multiplet** for each set of $q\bar{q}$ symmetries



It turns out that the proof of this is really interesting:

- We find that the irreducible representations in $g^{\otimes n_g}$ for varying N_c stand in a one to one, or one to zero correspondence to each other! (For each SU(3) multiplet there is an SU(5) version, but not vice versa.)
- Every multiplet in $g^{\otimes n_g}$ can be labeled in an N_c -independent way using the lengths of the *columns*. For example

$$\begin{array}{c} N_c-1 \\ 1 \\ \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \\ 8 \end{array} \otimes \begin{array}{c} N_c-1 \\ 1 \\ \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \\ 8 \end{array} = \begin{array}{c} N_c \\ \bullet \\ 1 \end{array} \oplus \begin{array}{c} N_c-1 \\ 1 \\ \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \\ 8 \end{array} \oplus \begin{array}{c} N_c-1 \\ 1 \\ \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \\ 8 \end{array} \oplus \begin{array}{c} N_c-2 \\ 1 \quad 1 \\ \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \\ 10 \end{array} \oplus \begin{array}{c} N_c-1 \\ N_c-1 \\ 2 \\ \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \\ 10 \end{array} \oplus \begin{array}{c} N_c-1 \\ N_c-1 \\ 1 \quad 1 \\ \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \\ 27 \end{array} \oplus \begin{array}{c} N_c-2 \\ \circ \\ 2 \end{array}$$

I have not seen this column notation elsewhere... have you?



Projecting out "old" multiplets

This would give us a way of constructing all projectors corresponding to "new" multiplets, if we knew how to project out all old multiplets.

In $g_1 \otimes g_2 \otimes g_3$, there are many 27-plets. How do we separate the various instance of the same multiplet?

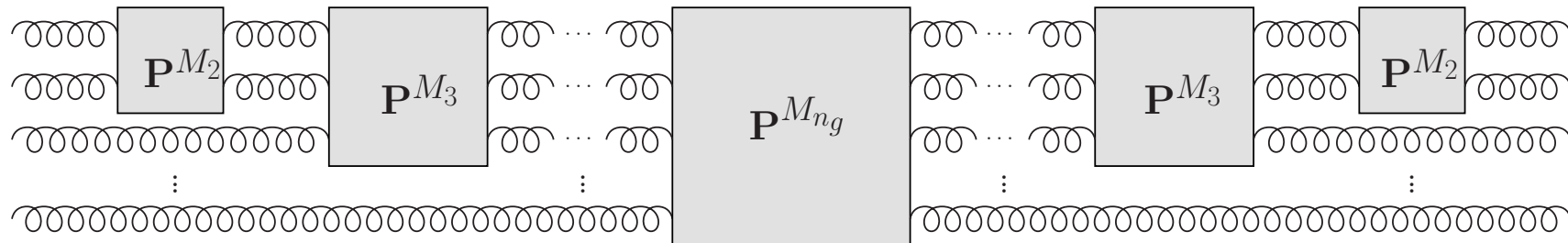


Projecting out "old" multiplets

This would give us a way of constructing all projectors corresponding to "new" multiplets, *if we knew how to project out all old multiplets.*

In $g_1 \otimes g_2 \otimes g_3$, there are many 27-plets. How do we separate the various instance of the same multiplet?

- *By the construction history!*



We make sure that the $n_g - \nu$ first gluons are in a given multiplet! Then the various instances are orthogonal as, at some point in the construction history, there was a different projector!
(More complicated for multiple occurrences...)



Construction of 3 gluon projectors

We start out by enumerating all projectors in $(8_1 \otimes 8_2) \otimes 8_3$

- Starting in a singlet, the result is trivial $1_{12} \otimes 8_3 = 8_{123}$
- If we start in an octet 8_{12} , $8_{12} \otimes 8_3$ is known from before:

$$\begin{array}{c} N_c-1 \\ 1 \\ \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \\ 8 \end{array} \otimes \begin{array}{c} N_c-1 \\ 1 \\ \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \\ 8 \end{array} = \begin{array}{c} N_c \\ \bullet \\ 1 \end{array} \oplus \begin{array}{c} N_c-1 \\ 1 \\ \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \\ 8 \end{array} \oplus \begin{array}{c} N_c-1 \\ 1 \\ \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \\ 8 \end{array} \oplus \begin{array}{c} N_c-2 \\ 1 \quad 1 \\ \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \\ 10 \end{array} \oplus \begin{array}{c} N_c-1 \\ N_c-1 \\ 2 \\ \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \\ \overline{10} \end{array} \oplus \begin{array}{c} N_c-1 \\ N_c-1 \\ 1 \quad 1 \\ \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \\ 27 \end{array} \oplus \begin{array}{c} N_c-2 \\ \circ \\ 0 \end{array}$$



- The 3g multiplets from (anti-) decuplets

$$\begin{array}{c}
 \begin{array}{c} N_c-2 \\ 1 \\ 1 \\ \hline 10 \end{array} \otimes \begin{array}{c} N_c-1 \\ 1 \\ \hline 8 \end{array} = \begin{array}{c} N_c-1 \\ 1 \\ \hline 8 \end{array} \oplus \begin{array}{c} N_c-2 \\ 1 \\ 1 \\ \hline 10 \end{array} \oplus \begin{array}{c} N_c-2 \\ 1 \\ 1 \\ \circ \\ (10) \end{array} \oplus \begin{array}{c} N_c-1 \\ N_c-1 \\ 1 \\ 1 \\ \hline 27 \end{array} \oplus \begin{array}{c} N_c-2 \\ \circ \\ 2 \end{array} \oplus \begin{array}{c} N_c-2 \\ 1 \\ 1 \\ 1 \\ \hline 35 \end{array} \\
 \oplus \begin{array}{c} N_c-1 \\ N_c-2 \\ 2 \\ 1 \\ \circ \\ 0 \end{array} \oplus \begin{array}{c} N_c-3 \\ 1 \\ 1 \\ 1 \\ \circ \\ 0 \end{array} \oplus \begin{array}{c} N_c-3 \\ 2 \\ 1 \\ \circ \\ 0 \end{array}
 \end{array}$$

$$\begin{array}{c}
 \begin{array}{c} N_c-1 \\ N_c-1 \\ 2 \\ \hline 10 \end{array} \otimes \begin{array}{c} N_c-1 \\ 1 \\ \hline 8 \end{array} = \begin{array}{c} N_c-1 \\ 1 \\ \hline 8 \end{array} \oplus \begin{array}{c} N_c-1 \\ N_c-1 \\ 2 \\ \hline 10 \end{array} \oplus \begin{array}{c} N_c-1 \\ N_c-1 \\ 2 \\ \circ \\ (10) \end{array} \oplus \begin{array}{c} N_c-1 \\ N_c-1 \\ 1 \\ 1 \\ \hline 27 \end{array} \oplus \begin{array}{c} N_c-2 \\ \circ \\ 2 \end{array} \oplus \begin{array}{c} N_c-1 \\ N_c-1 \\ N_c-1 \\ 2 \\ 1 \\ \hline 35 \end{array} \\
 \oplus \begin{array}{c} N_c-1 \\ N_c-2 \\ 2 \\ 1 \\ \circ \\ 0 \end{array} \oplus \begin{array}{c} N_c-1 \\ N_c-1 \\ N_c-1 \\ 3 \\ \circ \\ 0 \end{array} \oplus \begin{array}{c} N_c-1 \\ N_c-2 \\ 3 \\ \circ \\ 0 \end{array}
 \end{array}$$



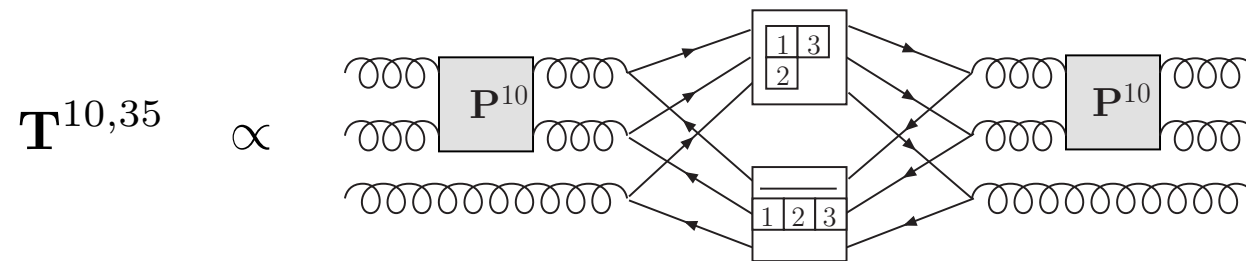
- The 3g multiplets from 27- and 0-plets

$$\begin{array}{ccccccc}
\begin{array}{c} N_{c-1} \\ N_{c-1} \\ 1 \\ 1 \\ \hline \square \quad \square \quad \square \quad \square \\ 27 \end{array} & \otimes & \begin{array}{c} N_{c-1} \\ 1 \\ \hline \square \quad \square \\ 8 \end{array} & = & \begin{array}{c} N_{c-1} \\ 1 \\ \hline \square \quad \square \\ 8 \end{array} & \oplus & \begin{array}{c} N_{c-2} \\ 1 \quad 1 \\ \hline \square \quad \square \quad \square \\ 10 \end{array} & \oplus & \begin{array}{c} N_{c-1} \\ N_{c-1} \\ 2 \\ \hline \square \quad \square \quad \square \quad \square \\ 10 \end{array} & \oplus & \begin{array}{c} N_{c-1} \\ N_{c-1} \\ 1 \quad 1 \\ \hline \square \quad \square \quad \square \quad \square \\ 27 \end{array} & \oplus & \begin{array}{c} N_{c-1} \\ N_{c-1} \\ 1 \quad 1 \\ \hline \square \quad \square \quad \square \quad \square \\ 27 \end{array} \\
\oplus & & & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus \\
\begin{array}{c} N_{c-1} \\ N_{c-1} \\ N_{c-1} \\ 2 \quad 1 \\ \hline \square \quad \square \quad \square \quad \square \quad \square \\ 35 \end{array} & & & & \begin{array}{c} N_{c-1} \\ N_{c-2} \\ 1 \quad 1 \quad 1 \\ \hline \square \quad \square \quad \square \quad \square \quad \square \\ 35 \end{array} & & & \oplus & & \begin{array}{c} N_{c-1} \\ N_{c-1} \\ N_{c-1} \\ 1 \quad 1 \quad 1 \\ \hline \square \quad \square \quad \square \quad \square \quad \square \\ 64 \end{array} & & & \oplus & & \begin{array}{c} N_{c-1} \\ N_{c-2} \\ 2 \quad 1 \\ \hline \circ \quad \circ \quad \circ \quad \circ \quad \circ \\ 0 \end{array}
\end{array}$$

$$\begin{array}{cccccccccccccccccccccccc}
\begin{array}{c} N_c-2 \\ \circ \\ \circ \end{array} & \otimes & \begin{array}{c} N_c-1 \\ \square \quad \square \\ \square \end{array} & = & \begin{array}{c} N_c-1 \\ \circ \\ (8) \end{array} & \oplus & \begin{array}{c} N_c-2 \\ \circ \\ (10) \end{array} & \oplus & \begin{array}{c} N_c-1 \\ \circ \\ (10) \end{array} & \oplus & \begin{array}{c} N_c-2 \\ \circ \end{array} & \oplus & \begin{array}{c} N_c-2 \\ \circ \end{array} & \oplus & \begin{array}{c} N_c-1 \\ \circ \\ N_c-2 \\ \circ \end{array} & \oplus & \begin{array}{c} N_c-3 \\ \circ \\ N_c-2 \\ \circ \end{array} & \oplus & \begin{array}{c} N_c-1 \\ \circ \\ N_c-2 \\ \circ \end{array}
\end{array}$$



- Construct projectors corresponding to “old” multiplets
- Construct the tensors which will give rise to “new” projectors



- From these, project out “old” multiplets

$$\mathbf{P}^{10,35} \propto \mathbf{T}^{10,35} - \sum_{m \subseteq 10 \otimes 8} \mathbf{P}^m \mathbf{T}^{10,35}$$

→ “new” projectors



- Calculations are done using a mathematica package, [ColorMath](#)
- Separate publication planned this autumn
- Intended to be an easy to use mathematica package for color summed calculations in QCD, $SU(N_c)$

```
In[1]:= Get ["/data/Documents/Annatjobb/Color/Mathematica/ColorMathv5.m"]
```

```
In[2]:= Amplitude = T t^{g} q^1_{q3} t^{g} q^4_{q2} + S t^{g} q^1_{q2} t^{g} q^4_{q3};
```

```
In[3]:= Amplitude Conjugate [Amplitude /. g -> g2] // CSimplify // Simplify
```

```
Out[3]= 
$$\frac{(-1 + N_c^2) (\text{Conjugate}[S] (-T + S N_c) + \text{Conjugate}[T] (-S + T N_c)) T_F^2}{N_c}$$

```



- In this way we have constructed the projection operators onto irreducible subspaces for $3g \rightarrow 3g$
- There are 51 projectors, reducing to 29 for $SU(3)$
- From these we have constructed an **orthogonal** (normalized) basis for the $6g$ space, by letting any instance of a given multiplet go to any other instance of the same multiplet. For general N_c there are 265 basis vectors. Crossing out tensors that do not appear for $N_c = 3$, we get a **minimal** basis with 145 basis vectors.

There's also a reduction from charge conjugation



Number of projection operators and basis vectors

In general, for many partons the size of the vector space is much smaller for $N_c = 3$, compared to for $N_c \rightarrow \infty$

Case	Projectors $N_c = 3$	Projectors $N_c = \infty$	Vectors $N_c = 3$	Vectors $N_c = \infty$
$2g \rightarrow 2g$	6	7	8	9
$3g \rightarrow 3g$	29	51	145	265
$4g \rightarrow 4g$	166	513	3 598	14 833
$5g \rightarrow 5g$	1 002	6 345	107 160	1 334 961

Number of projection operators and basis vectors for $N_g \rightarrow N_g$

gluons *without* imposing projection operators and vectors to appear in charge conjugation invariant combinations



- The **size** of the vector spaces asymptotically grows as an **exponential** in the number of gluons/ $q\bar{q}$ -pairs for **finite** N_c
- For **general** N_c the basis size grows as a **factorial**

$$N_{\text{vec}}[n_q, N_g] = N_{\text{vec}}[n_q, N_g - 1](N_g - 1 + n_q) + N_{\text{vec}}[n_q, N_g - 2](N_g - 1)$$

where

$$N_{\text{vec}}[n_q, 0] = n_q!$$

$$N_{\text{vec}}[n_q, 1] = n_q n_q!$$

As the multiplet basis also is orthogonal it has the potential to very significantly speed up exact calculations in QCD!



Processes with quarks

- We can also construct bases for processes with quarks using the gluon projection operators. To see this we note that a $q\bar{q}$ -pair may either be in an octet – in which case we may replace it with a gluon, or in a singlet – in which case we enforce this and use the gluon basis for one less gluon
- In general, having the $n_g \rightarrow n_g$ projectors we can easily get the bases for up to $2n_g + 1$ gluons plus $q\bar{q}$ pairs
- Knowing how to construct the gluon projection operators in general, we thus know how to construct the basis vectors for any number and kind of partons and any order in perturbation theory!



Conclusions

- We have outlined a general recipe for construction of minimal orthogonal multiplet based bases for any QCD process, arXiv:1207.0609
- On the way we found an N_c -independent labeling of the multiplets in $g^{\otimes n_g}$, and a one to one, or one to zero, correspondence between these for various N_c
- Number of basis vectors grows only exponentially for $N_c = 3$
- This has the potential to very significantly speed up exact calculations in the color space of $SU(N_c)$



...and outlook

- However, in order to use this in an optimized way, we need to understand how to sort QCD amplitudes in this basis in an efficient way
- ...also, a lot of implementational work remains



Backup: Some example projectors

$$\mathbf{P}_{g_1 g_2 g_3 g_4 g_5 g_6}^{8a,8a} = \frac{1}{T_R^2} \frac{1}{4N_c^2} i f_{g_1 g_2 i_1} i f_{i_1 g_3 i_2} i f_{g_4 g_5 i_3} i f_{i_3 g_6 i_2}$$

$$\mathbf{P}_{g_1 g_2 g_3 g_4 g_5 g_6}^{8s,27} = \frac{1}{T_R} \frac{N_c}{2(N_c^2 - 4)} d_{g_1 g_2 i_1} \mathbf{P}_{i_1 g_3 i_2 g_6}^{27} d_{i_2 g_4 g_5}$$

$$\mathbf{P}_{g_1 g_2 g_3 g_4 g_5 g_6}^{27,8} = \frac{4(N_c + 1)}{N_c^2(N_c + 3)} \mathbf{P}_{g_1 g_2 i_1 g_3}^{27} \mathbf{P}_{i_1 g_6 g_4 g_5}^{27}$$

$$\begin{aligned} \mathbf{P}_{g_1 g_2 g_3 g_4 g_5 g_6}^{27,64=c111c111} &= \frac{1}{T_R^3} \mathbf{T}_{g_1 g_2 g_3 g_4 g_5 g_6}^{27,64} - \frac{N_c^2}{162(N_c + 1)(N_c + 2)} \mathbf{P}_{g_1 g_2 g_3 g_4 g_5 g_6}^{27,8} \\ &- \frac{N_c^2 - N_c - 2}{81N_c(N_c + 2)} \mathbf{P}_{g_1 g_2 g_3 g_4 g_5 g_6}^{27,27s} \end{aligned}$$



Backup: Three gluon multiplets

$SU(3)$ dim	1	8	10	$\overline{10}$	27	0
Multiplet	c0c0	c1c1	c11c2	c2c11	c11c11	c2c2
	$((45)^{8s}_6)^1$	$2 \times ((45)^{8s}_6)^{8s \text{ or } a}$	$((45)^{8s}_6)^{10}$	$((45)^{8s}_6)^{\overline{10}}$	$((45)^{8s}_6)^{27}$	$((45)^{8s}_6)^0$
	$((45)^{8a}_6)^1$	$2 \times ((45)^{8a}_6)^{8s \text{ or } a}$	$((45)^{8a}_6)^{10}$	$((45)^{8a}_6)^{\overline{10}}$	$((45)^{8a}_6)^{27}$	$((45)^{8a}_6)^0$
		$((45)^{10}_6)^8$	$((45)^{10}_6)^{10}$	$((45)^{\overline{10}}_6)^{\overline{10}}$	$((45)^{10}_6)^{27}$	$((45)^{10}_6)^0$
		$((45)^{\overline{10}}_6)^8$	$((45)^{10}_6)^{10}$	$((45)^{\overline{10}}_6)^{\overline{10}}$	$((45)^{\overline{10}}_6)^{27}$	$((45)^{\overline{10}}_6)^0$
		$((45)^{27}_6)^8$	$((45)^{27}_6)^{10}$	$((45)^{27}_6)^{\overline{10}}$	$((45)^{27}_6)^{27}$	$((45)^0_6)^0$
		$((45)^0_6)^8$	$((45)^0_6)^{10}$	$((45)^0_6)^{\overline{10}}$	$((45)^{27}_6)^{27}$	$((45)^0_6)^0$
$SU(3)$ dim	64	35	$\overline{35}$	0		
Multiplet	c111c111	c111c21	c21c111	c21c21		
	$((45)^{27}_6)^{64}$	$((45)^{10}_6)^{35}$	$((45)^{\overline{10}}_6)^{\overline{35}}$	$((45)^{10}_6)^{c21c21}$		
		$((45)^{27}_6)^{35}$	$((45)^{27}_6)^{\overline{35}}$	$((45)^{\overline{10}}_6)^{c21c21}$		
				$((45)^{27}_6)^{c21c21}$		
				$((45)^0_6)^{c21c21}$		
$SU(3)$ dim	0	0	0	0	0	
Multiplet	c111c3	c3c111	c21c3	c3c21	c3c3	
	$((45)^{10}_6)^{c111c3}$	$((45)^{\overline{10}}_6)^{c3c111}$	$((45)^{10}_6)^{c21c3}$	$((45)^{\overline{10}}_6)^{c3c21}$	$((45)^0_6)^{c3c3}$	
			$((45)^0_6)^{c21c3}$	$((45)^0_6)^{c3c21}$		

Multiplets for $g_4 \otimes g_5 \otimes g_6$



The importance of Hermitian projectors

$$\begin{aligned}
 \mathbf{P}_Y^{6,8} &= \frac{4}{3} \begin{array}{c} \text{Diagram 1: White box on left, black box on right. Three horizontal lines enter from the left. The top two lines cross each other and then cross the black box. The bottom line goes straight through the black box. Three lines exit to the right.} \end{array}, \quad \mathbf{P}^{6,8} = \frac{4}{3} \begin{array}{c} \text{Diagram 2: White box on left, black box in middle, white box on right. Three horizontal lines enter from the left. The top two lines cross each other and then cross the black box. The bottom line goes straight through the black box. Three lines exit to the right.} \end{array} \\
 \mathbf{P}_Y^{\bar{3},8} &= \frac{4}{3} \begin{array}{c} \text{Diagram 3: White box on right, black box on left. Three horizontal lines enter from the left. The top two lines cross each other and then cross the black box. The bottom line goes straight through the black box. Three lines exit to the right.} \end{array}, \quad \mathbf{P}^{\bar{3},8} = \frac{4}{3} \begin{array}{c} \text{Diagram 4: Black box on left, white box in middle, black box on right. Three horizontal lines enter from the left. The top two lines cross each other and then cross the white box. The bottom line goes straight through the white box. Three lines exit to the right.} \end{array}
 \end{aligned}$$

The standard Young projection operators $\mathbf{P}_Y^{6,8}$ and $\mathbf{P}_Y^{\bar{3},8}$ compared to their hermitian versions $\mathbf{P}^{6,8}$ and $\mathbf{P}^{\bar{3},8}$.

Clearly $\mathbf{P}^{6,8\dagger} \mathbf{P}^{\bar{3},8} = \mathbf{P}^{6,8} \mathbf{P}^{\bar{3},8} = 0$. However, as can be seen from the symmetries, $\mathbf{P}_Y^{6,8\dagger} \mathbf{P}_Y^{\bar{3},8} \neq 0$.



Backup: First occurrence




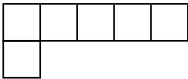
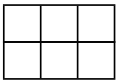
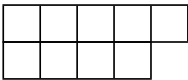
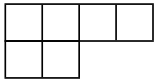
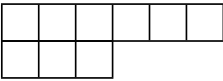
n_f	0	1	2	3
$SU(3)$	• = 			
Young diagrams				
				

Table 1: Examples of $SU(3)$ Young diagrams sorted according to their first occurrence n_f .



Backup: Gluon exchange

A gluon exchange in this basis “directly” i.e. without using scalar products gives back a linear combination of (at most 4) basis tensors

$$\begin{aligned}
 & \text{Diagram 1} = 2 \text{ Diagram 2} - 2 \text{ Diagram 3} \\
 & \text{Fierz} = \text{Diagram 4} - \text{Diagram 5} + \text{canceling } N_c\text{-suppressed terms} \\
 & \text{Fierz } \frac{1}{2} = \frac{1}{2} \text{ Diagram 6} - \frac{1}{2} \text{ Diagram 7} + \text{canceling } N_c\text{-suppressed terms} \\
 & = \frac{N_c}{2} \text{ Diagram 8} - 0
 \end{aligned}$$

- N_c -enhancement possible only for near by partons
 \rightarrow only “color neighbors” radiate in the $N_c \rightarrow \infty$ limit

