Electromagnetic waves

In regions without any sources ("far away") Maxwell's eqn's read

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\times\vec{E} + \frac{\partial}{\partial t}\vec{B} = 0$$
$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} - \mu_0\varepsilon_0\frac{\partial}{\partial t}\vec{E} = 0$$

taking the curl of Faraday's law gives

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) + \frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B}) = 0$$

$$\vec{\nabla}\underbrace{(\vec{\nabla}\cdot\vec{E})}_{=0} - \vec{\nabla}^2\vec{E} + \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\vec{E} = 0$$

$$\Rightarrow \quad \vec{\nabla}^2\vec{E} = \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\vec{E} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}$$

or using $\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right)$; $\partial^2\vec{E} = 0$

similarly $\quad \vec{\nabla}^2\vec{B} = \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\vec{B}$

– the wave eqn in 3 space dimensions with plane wave solution

wave-vector

$$\vec{\tilde{E}}(\vec{r},t) = \vec{\tilde{E}}_{\pm}\, e^{i(\vec{k}\cdot\vec{r}\pm\omega t)}$$

angular frequency

complex amplitude

check: $\vec{\nabla}^2\vec{\tilde{E}} = -\vec{k}^2\,\vec{\tilde{E}}$

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{\tilde{E}} = -\frac{1}{c^2}\omega^2\,\vec{\tilde{E}}$$

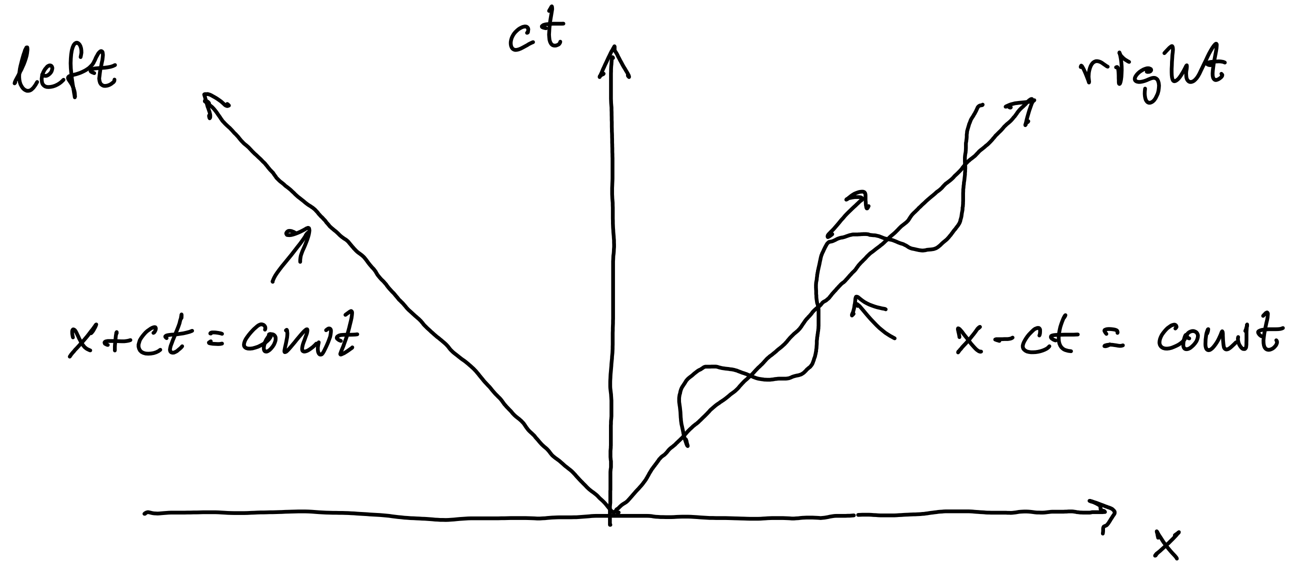
true if $|\vec{k}| = k = \frac{\omega}{c} = \frac{2\pi}{\lambda}$ ← wave-length

wave-number   propagation speed

The wave moves along so called light-cones

Choose coordinate system such that the wave moves along the x-axis ($\vec{k} = (k,0,0)$).

Then we have $\quad kx - \omega t = k(x - ct)$



note: the solutions are complex (simplifies the formalism) - the physical fields are given by the real parts, explicitly

$$Re(\vec{\tilde{E}}) = Re\left(|\vec{\tilde{E}}| \, e^{i(\vec{k}\cdot\vec{r} - \omega t + \delta)} \right) \hat{n}$$

$$= |\vec{\tilde{E}}| \cos(\vec{k}\cdot\vec{r} - \omega t + \delta) \, \hat{n}$$

phase

note: in the general case a wave has several frequencies - a wave-packet

$$\vec{\tilde{E}} = \int \vec{\tilde{E}}_{\pm}(\vec{k}) \, e^{i(\vec{k}\cdot\vec{r} \pm \omega t)} \, d^3\vec{k}$$

fourier components

but thanks to the linearity of Maxwell's equ's it is often enough to consider one frequency at a time note: a plane wave extends to infinity in the transverse directions

Writing the Laplace operator in spherical coordinates (r, θ, φ) and assuming spherical symmetry we have

$$\vec{\nabla}^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} f\right) = \frac{1}{r^2}\, 2r \frac{\partial}{\partial r} f + \frac{\partial^2}{\partial r^2} f$$

$$= \frac{1}{r}\frac{\partial^2}{\partial r^2}(r f)$$

which gives

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r \tilde{\vec{E}}) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tilde{\vec{E}}$$

$$\Rightarrow \quad \frac{\partial^2}{\partial r^2}(r \tilde{\vec{E}}) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(r \tilde{\vec{E}})$$

with solutions

$$r \tilde{\vec{E}}(\vec{r}, t) = \tilde{\vec{E}}_{\pm}\, e^{i(kr \pm \omega t)}$$

or

$$\tilde{\vec{E}}(\vec{r}, t) = \tilde{\vec{E}}_{\pm}\, \frac{e^{i(kr \pm \omega t)}}{r}$$

spherical waves moving $\left\{ \begin{array}{l} \text{into } (+) \\ \text{out from } (-) \end{array} \right\}$ the origin

will come back to this when we consider dipole radiation (and scattering)

A wave propagates in the $\hat{k} = \frac{\vec{k}}{k}$ direction

Assuming that

$$\vec{\tilde{E}} = \tilde{E}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\, \hat{n}$$

Gauss' law $\vec{\nabla}\cdot\vec{E} = 0$ gives $\hat{k}\cdot\hat{n} = 0$

Faraday's law $\vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}\vec{B}$ gives

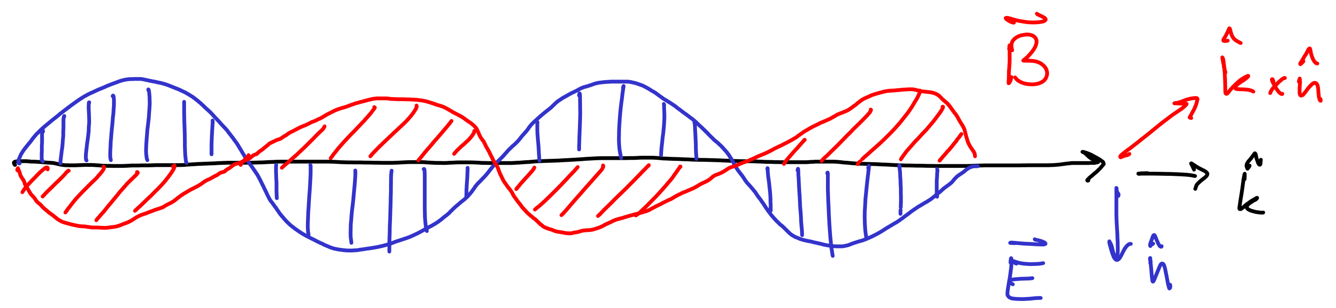
$$i\,\tilde{E}\, k\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\, \hat{k}\times\hat{n} = i\omega\, \vec{\tilde{B}}$$

$$\Rightarrow \vec{\tilde{B}} = \underbrace{\frac{k}{\omega}}_{\frac{1}{c}}\, \tilde{E}\, \hat{k}\times\hat{n}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

note: $\vec{B}\cdot\hat{k} = 0$, $\vec{B}\cdot\vec{E} = 0$

$$|\vec{B}| = \frac{1}{c}|\vec{E}|$$

note: $\vec{B}$ - field contains no extra information (same phase, speed)



called linearly polarised wave

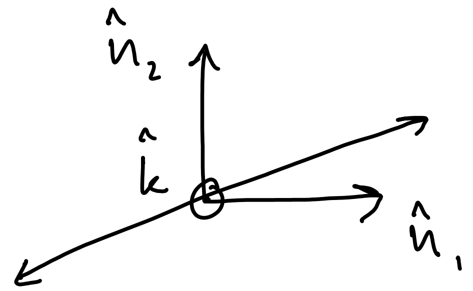
$(\hat{k}, \hat{n}, \hat{k}\times\hat{n})$ right-handed system

In the general case we can write $\vec{E}$ as

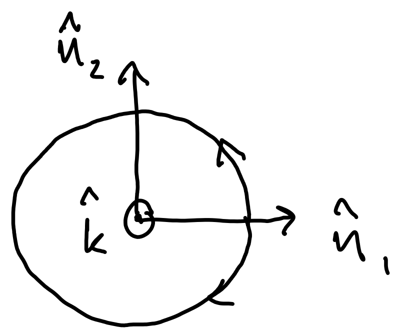
$$\tilde{\vec{E}} = \hat{n}_1 |\tilde{E}_1| e^{i(\vec{k}\cdot\vec{r}-\omega t+\delta_1)} + \hat{n}_2 |\tilde{E}_2| e^{i(\vec{k}\cdot\vec{r}-\omega t+\delta_2)}$$

where $\hat{k}\cdot\hat{n}_1 = \hat{k}\cdot\hat{n}_2 = 0$ and preferentially $\hat{n}_1\cdot\hat{n}_2 = 0$

If $\delta_1 = \delta_2$ then we have linear polarisation
" $\delta_1 = \delta_2 \pm \frac{\pi}{2}$ and $|\tilde{E}_1| = |\tilde{E}_2|$, circular — " —



linear



circular

Energy transported by plane wave:

energy densities:

$$u_E = \frac{1}{2}\varepsilon_0\left[Re(\tilde{\vec{E}})\right]^2 , \quad u_M = \frac{1}{2}\frac{1}{\mu_0}\left[Re(\tilde{\vec{B}})\right]^2$$

$$|\vec{B}| = \frac{1}{c}|\vec{E}| , \quad \frac{1}{c^2} = \mu_0\varepsilon_0$$

$$\Rightarrow \quad u_M = u_E = \frac{1}{2}\varepsilon_0 |\tilde{E}|^2 \cos^2(\vec{k}\cdot\vec{r} - \omega t + \delta)$$

$$u = u_E + u_M = 2u_E$$

Poynting's vector: $[\hat{n} \times (\hat{k} \times \hat{n})) = \hat{k}]$

$$\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B}) = c\underbrace{\varepsilon_0 |\tilde{E}|^2 \cos^2(\vec{k}\cdot\vec{r} - \omega t + \delta)}_{u}\,\hat{k}$$

We also get the time averages: $\langle \cos^2(\,)\rangle = \frac{1}{2}$

$$\langle u \rangle = \frac{1}{2}\varepsilon_0 |\tilde{E}|^2$$

$$\langle \vec{S} \rangle = \frac{1}{2}c\varepsilon_0 |\tilde{E}|^2 \hat{z} = c\langle u \rangle \hat{z}$$

and the intensity, I, the average power per unit area

$$I = |\langle \vec{S} \rangle| = c\langle u \rangle = \frac{1}{2}c\varepsilon_0 |\tilde{E}|^2$$

note: the time-average can be written as

$$\langle u_E \rangle = \langle \frac{1}{2}\varepsilon_0 \vec{E}^2 \rangle =$$

$$\left[\vec{E}^2 = \left(Re(\tilde{E}e^{i(\vec{k}\cdot\vec{r} - \omega t)})\right)^2 = \frac{1}{4}\left(\tilde{E}e^{i(\vec{k}\cdot\vec{r} - \omega t)} + \tilde{E}^* e^{-i(\vec{k}\cdot\vec{r} - \omega t)}\right)^2\right]$$

$$= \frac{1}{2}\varepsilon_0 \frac{1}{4}\langle \tilde{E}^2 e^{-2i\omega t} + \tilde{E}^{*2} e^{2i\omega t} + 2\tilde{E}\tilde{E}^* \rangle$$

$$= \frac{1}{4}\varepsilon_0 \tilde{E}\tilde{E}^* = \frac{1}{4}\varepsilon_0 |\tilde{E}|^2$$

Wave-propagation in linear media:

Maxwells eqn's in matter with no free charge and no free currents

$$\vec\nabla \cdot \vec D = 0 \qquad \vec\nabla \times \vec E = -\frac{\partial}{\partial t} \vec B$$
$$\vec\nabla \cdot \vec B = 0 \qquad \vec\nabla \times \vec H = \frac{\partial}{\partial t} \vec D$$

for linear (+homogeneous, isotropic) media

$$\vec D = \underset{\uparrow\, \varepsilon_0 \varepsilon_r}{\varepsilon} \vec E \qquad \vec B = \underset{\uparrow\, \mu_0 \mu_r}{\mu} \vec H$$

we get same eqns as in vacuum:

$$\vec\nabla \cdot \vec E = 0 \qquad \vec\nabla \times \vec E = -\frac{\partial}{\partial t} \vec B$$
$$\vec\nabla \cdot \vec B = 0 \qquad \vec\nabla \times \vec B = \underbrace{\mu \varepsilon}_{\mu_r \varepsilon_r \mu_0 \varepsilon_0} \frac{\partial}{\partial t} \vec E$$

only difference - speed of propagation

$$v = \frac{1}{\sqrt{\mu \varepsilon}} = \frac{1}{\sqrt{\mu_r \varepsilon_r}} \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \frac{c}{\underset{\uparrow \text{ index of refraction}}{n}}$$

$$k v = \omega$$

as before the energy densities are

$$u_E = \frac{1}{2} \varepsilon \vec E^2 \qquad u_M = \frac{1}{2} \frac{1}{\mu} \vec B^2 = \frac{1}{2} \mu \vec H^2$$

and Poynting vector and the intensity

$$\vec S = \frac{1}{\mu} (\vec E \times \vec B) = \vec E \times \vec H \quad , \quad |\vec B| = \frac{1}{v} |\vec E|$$

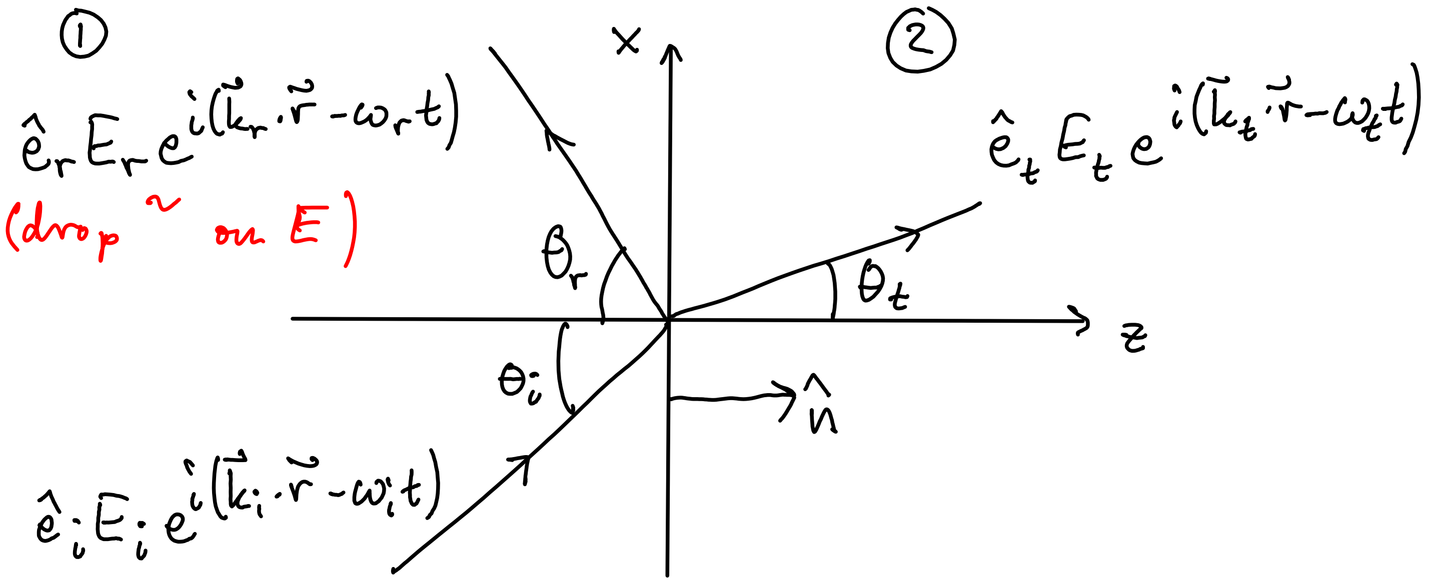
$$I = |\langle \vec S \rangle| = \frac{1}{2} \frac{1}{\mu v} |\tilde E|^2 = \frac{1}{2} \varepsilon v |\tilde E|^2 = \frac{1}{2} \sqrt{\frac{\varepsilon}{\mu}} |\tilde E|^2 = \frac{1}{2} \frac{1}{Z} |\tilde E|^2$$

$$\left[Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \mu_0 c = \frac{1}{\varepsilon_0 c} \approx 377 \text{ ohm} \quad , \quad P = UI = \frac{U^2}{R} \right]$$

Reflection and transmission:

Consider light going from an optically thinner (1) medium to an optically thicker (2), i.e. $n_i < n_2$

Plane waves for incoming, reflected and transmitted waves:



The electric fields on the two sides are

$$\vec{E}_1 = \hat{e}_r E_r e^{i(\vec{k}_r \cdot \vec{r} - \omega_r t)} + \hat{e}_i E_i e^{i(\vec{k}_i \cdot \vec{r} - \omega_i t)}$$
$$\vec{E}_2 = \hat{e}_t E_t e^{i(\vec{k}_t \cdot \vec{r} - \omega_t t)}$$

Boundary conditions:

$$D_1^\perp - D_2^\perp = \sigma_f = 0 \quad \Rightarrow \quad \varepsilon_1 E_1^\perp = \varepsilon_2 E_2^\perp$$
$$B_1^\perp - B_2^\perp = 0 \quad \Rightarrow \quad B_1^\perp = B_2^\perp$$
$$\vec{E}_1^{\parallel} - \vec{E}_2^{\parallel} = 0 \quad \Rightarrow \quad \vec{E}_1^{\parallel} = \vec{E}_2^{\parallel}$$
$$\vec{H}_1^{\parallel} - \vec{H}_2^{\parallel} = \vec{K}_f \times \hat{n} = 0 \quad \Rightarrow \quad \frac{1}{\mu_1} \vec{B}_1^{\parallel} = \frac{1}{\mu_2} \vec{B}_2^{\parallel}$$

(same for real and imag. parts)

these conditions apply for all t, x, y

$$\Rightarrow \begin{cases} \omega_i = \omega_r = \omega_t = \omega \\ (\vec{k}_i \cdot \vec{r})_{z=0} = (\vec{k}_r \cdot \vec{r})_{z=0} = (\vec{k}_t \cdot \vec{r})_{z=0} \end{cases}$$

in other words all $\vec{k}$ are in the same plane
- choose the xz-plane

$$\Rightarrow \quad k_i \sin\theta_i = k_r \sin\theta_r = k_t \sin\theta_t$$

(x-components are the same)

since $v_i = v_r = v_1 = \frac{c}{n_1}$, $v_t = v_2 = \frac{c}{n_2}$ and $k v = \omega$

we also get

$$\Big[\; v_i = v_r \Rightarrow k_i = k_r \Rightarrow \theta_i = \theta_r$$

$$\frac{\omega}{v_i} \sin\theta_i = \frac{\omega}{v_t} \sin\theta_t \Rightarrow \frac{\omega n_1}{c} \sin\theta_i = \frac{\omega n_2}{c} \sin\theta_t \;\Big]$$

$$\begin{cases} \theta_i = \theta_r & \text{law of reflection} \\ n_1 \sin\theta_i = n_2 \sin\theta_t & \text{law of refraction} \end{cases}$$

note: these do <u>not</u> depend on the dynamics!

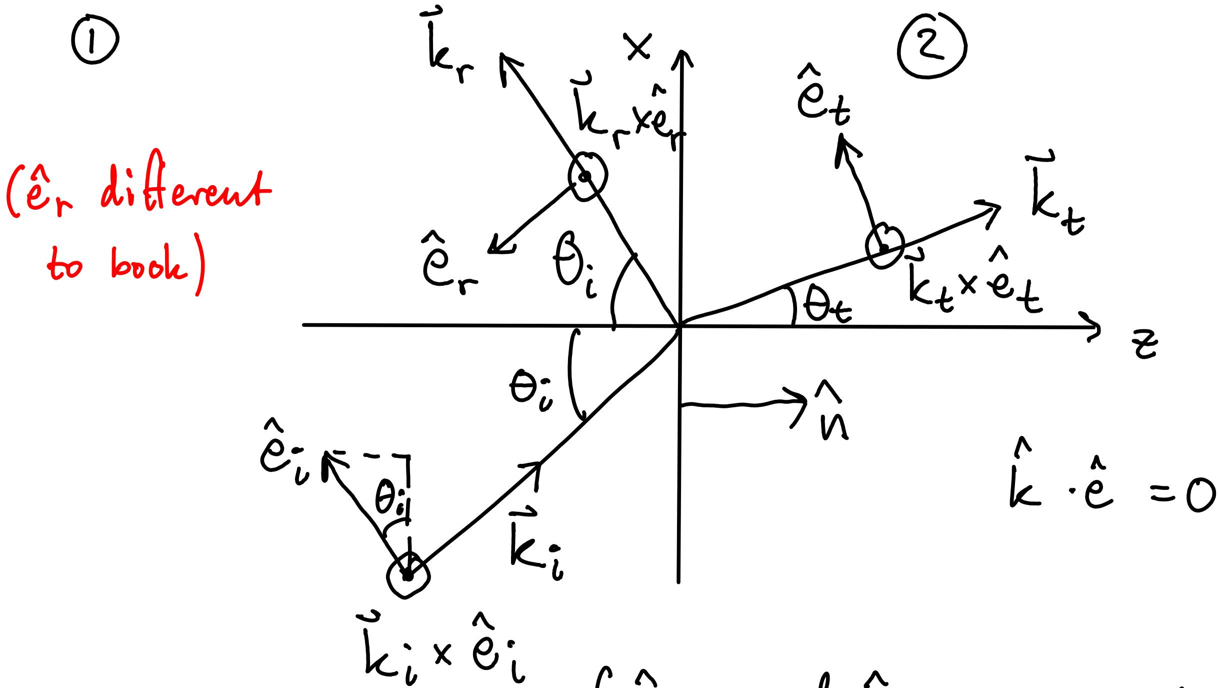
The magnetic fields:

$$\vec{B}_1 = \frac{1}{v_1}\left(E_r e^{i(\vec{k}_r \cdot \vec{r} - \omega t)} \hat{k}_r \times \hat{e}_r + E_i e^{i(\vec{k}_i \cdot \vec{r} - \omega t)} \hat{k}_i \times \hat{e}_i \right)$$

$$\vec{B}_2 = \frac{1}{v_2} E_t e^{i(\vec{k}_t \cdot \vec{r} - \omega t)} \hat{k}_t \times \hat{e}_t$$

To proceed we have to specify the polarisation:

Consider first the case when $\hat{e}_i$ is in the xz-plane – called transverse magnetic (TM) polarisation



boundary cond. $\Rightarrow \begin{cases} \hat{e}_r \text{ and } \hat{e}_t \text{ in } xz\text{-plane} \\ \vec{E}^{\parallel} \text{ and } \vec{H}^{\parallel} \text{ continuous} \end{cases}$

$$\begin{cases} E_i \cos\theta_i - E_r \cos\theta_i = E_t \cos\theta_t \\ \dfrac{1}{\mu_1 v_1} E_i + \dfrac{1}{\mu_1 v_1} E_r = \dfrac{1}{\mu_2 v_2} E_t \end{cases}$$

$$|\vec{B}| = \frac{1}{v} |\vec{E}|$$

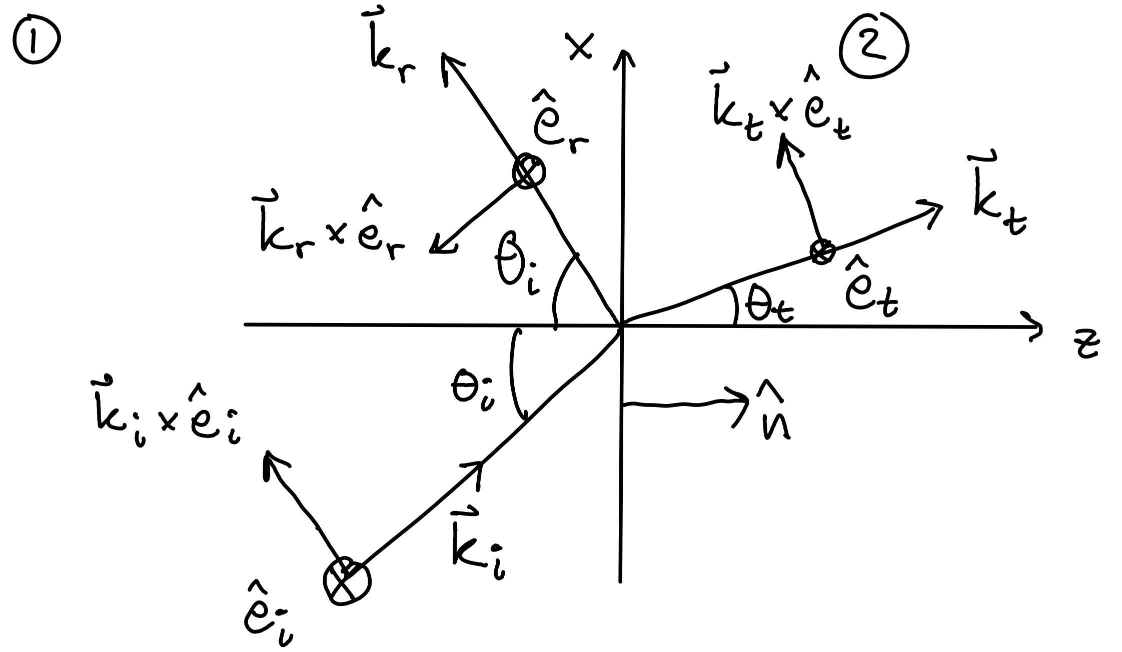
$$|\vec{H}| = \frac{1}{\mu} |\vec{B}|$$

using $v = \frac{c}{n}$ the solutions are

$$\begin{cases} \dfrac{E_t}{E_i} = \dfrac{2\mu_2 n_1 \cos\theta_i}{\mu_1 n_2 \cos\theta_i + \mu_2 n_1 \cos\theta_t} \\ \dfrac{E_r}{E_i} = \dfrac{\mu_1 n_2 \cos\theta_i - \mu_2 n_1 \cos\theta_t}{+} \end{cases}$$

Fresnel's equ's

Consider next the case when $\hat{e}_i = -\hat{y}$ - called transverse electric (TE) polarisation (problem 9.17)



again $\vec{E}^{\parallel}$ and $\vec{H}^{\parallel}$ are continuous giving

$$\frac{E_t}{E_i} = \frac{2\mu_2 n_1 \cos\theta_i}{\mu_2 n_1 \cos\theta_i + \mu_1 n_2 \cos\theta_t}$$

$$\frac{E_r}{E_i} = \frac{\mu_2 n_1 \cos\theta_i - \mu_1 n_2 \cos\theta_t}{+}$$

In both cases normal incidence $\theta_i = 0$ gives

$$\frac{E_t}{E_i} = \frac{2\mu_2 n_1}{\mu_2 n_1 + \mu_1 n_2}$$

$$\frac{E_r}{E_i} = \frac{\mu_1 n_2 - \mu_2 n_1}{+}$$ (up to a sign depending on how $\hat{e}_r$ is defined)

if instead $\cos\theta_i \approx 0$ we get

$$E_t \approx 0$$

$$\frac{E_r}{E_i} = -1$$ phase shifted with π

For TM waves the reflected wave vanishes if

$$\mu_1 n_2 \cos\theta_i = \mu_2 n_1 \cos\theta_t = \mu_2 n_1 \sqrt{1 - \frac{n_1^2}{n_2^2}\sin^2\theta_i}$$

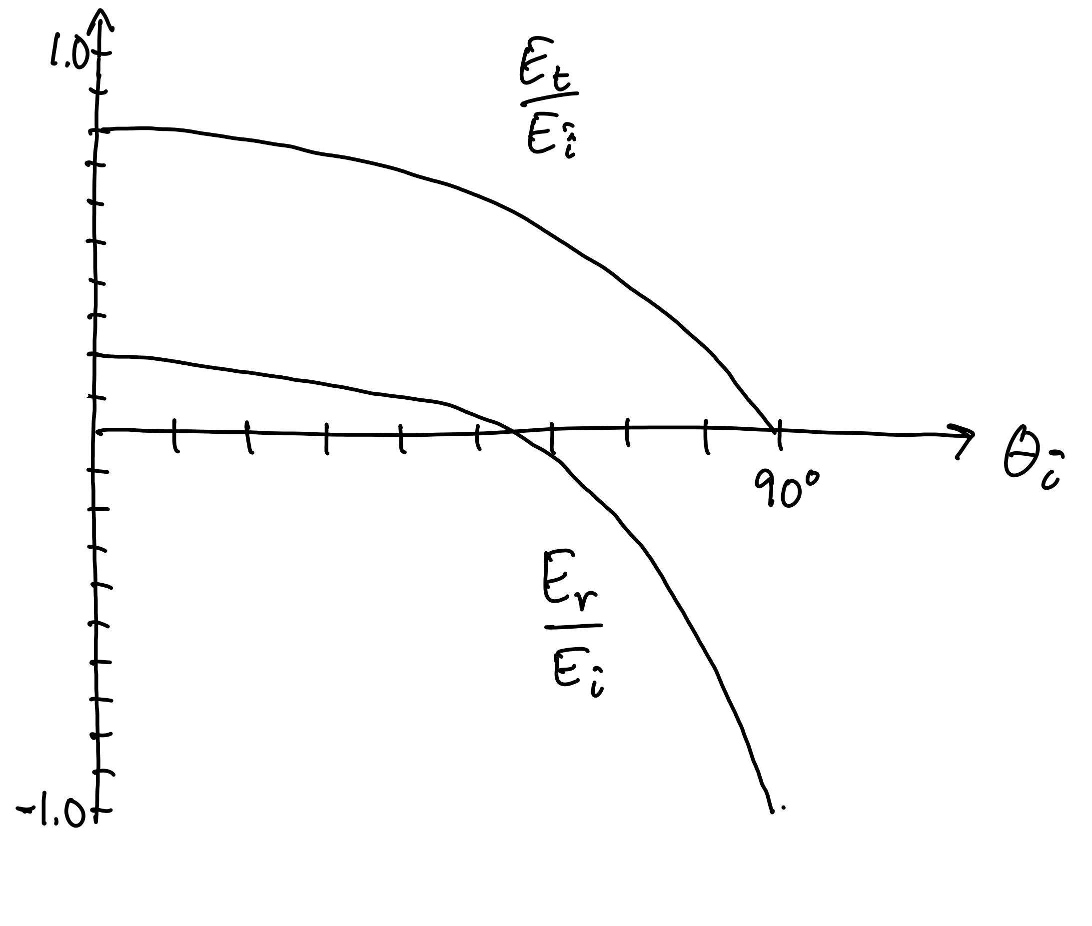
$$[n_1 \sin\theta_i = n_2 \sin\theta_t]$$

with the solution (assuming $\mu_1 = \mu_2$)

$$\tan\theta_B = \frac{n_2}{n_1} \qquad \text{Brewster angle}$$

(the idea behind Polaroid shades)

Example: $\mu_1 = \mu_2 = 1$, $n_2 = 1.5$, $n_1 = 1$ gives for TM polarisation



"Energy conservation"

To look at the energy-transport we need Poynting's vector for the two sides

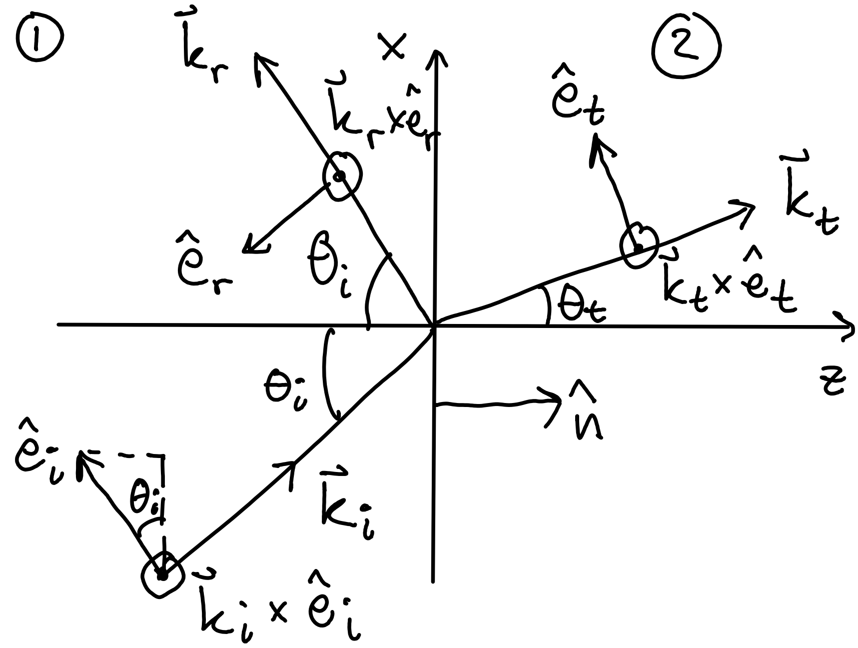
$$\vec{S}_{1,2} = \vec{E}_{1,2} \times \vec{H}_{1,2} \quad , \quad \langle \vec{S}_{1,2} \rangle = \frac{1}{2} \mathrm{Re}\left(\vec{E}_{1,2} \times \vec{H}^*_{1,2} \right)$$

$$\vec{E}_1 = \vec{E}_i + \vec{E}_r$$

$$\vec{H}_1 = \vec{H}_i + \vec{H}_r$$

$$\vec{E}_2 = \vec{E}_t$$

$$\vec{H}_2 = \vec{H}_t$$



Consider TM-polarisation

$$\vec{E}_i = \hat{e}_i E_i e^{i(\vec{k}_i \cdot \vec{r} - \omega t)} \quad , \quad \hat{e}_i = \hat{x}\cos\theta_i - \hat{z}\sin\theta_i$$

$$\vec{H}_i = \frac{1}{\mu_1 v_1} \hat{k}_i \times \hat{e}_i E_i e^{i(\vec{k}_i \cdot \vec{r} - \omega t)} \quad , \quad \hat{k}_i \times \hat{e}_i = \hat{y}$$

similarly for reflected wave but

$$\hat{e}_r = -\hat{x}\cos\theta_i - \hat{z}\sin\theta_i \quad , \quad \hat{k}_r \times \hat{e}_r = \hat{y}$$

$$\Rightarrow \langle \vec{S}_1 \rangle = \frac{1}{2\mu_1 v_1} \mathrm{Re}\Big[\underbrace{(\hat{e}_i E_i + \hat{e}_r E_r) \times \hat{y}}_{\hat{z}(E_i - E_r)\cos\theta_i + \hat{x}(E_i + E_r)\sin\theta_i} (E_i^* + E_r^*) \Big]$$

$$\therefore \langle \vec{S}_1 \rangle \cdot \hat{z} = \frac{1}{2\mu_1 v_1} \mathrm{Re}\Big[(E_i^* + E_r^*)(E_i - E_r) \Big] \cos\theta_i =$$

$$= \frac{1}{2\mu_1 v_1} |E_i|^2 \cos\theta_i - \frac{1}{2\mu_1 v_1} |E_r|^2 \cos\theta_i = \left(\langle \vec{S}_i \rangle + \langle \vec{S}_r \rangle \right) \cdot \hat{z}$$

for the transmitted wave we have

$$\langle \vec{S}_2 \rangle \cdot \hat{z} = \langle \vec{S}_t \rangle \cdot \hat{z} = \frac{1}{2\mu_2 v_2} |E_t|^2 \cos\theta_t$$

the boundary conditions for $\vec{E}^{\parallel}$ and $\vec{H}^{\parallel}$

$$\begin{cases} E_i \cos\theta_i - E_r \cos\theta_i = E_t \cos\theta_t \\ \frac{1}{\mu_1 v_1} E_i^* + \frac{1}{\mu_1 v_1} E_r^* = \frac{1}{\mu_2 v_2} E_t^* \end{cases}$$

then tells us that

$$\langle \vec{S}_1 \rangle \cdot \hat{z} = \langle \vec{S}_2 \rangle \cdot \hat{z}$$

$$\Rightarrow \underbrace{\frac{1}{2\mu_1 v_1} |E_i|^2 \cos\theta_i}_{I_i} = \underbrace{\frac{1}{2\mu_1 v_1} |E_r|^2 \cos\theta_i}_{I_r} + \underbrace{\frac{1}{2\mu_2 v_2} |E_t|^2 \cos\theta_t}_{I_t}$$

$$\therefore \quad I_i = I_r + I_t$$

$$\frac{I_r}{I_i} + \frac{I_t}{I_i} = \underbrace{\frac{|E_r|^2}{|E_i|^2}}_{R} + \underbrace{\overbrace{\frac{\mu_1 v_1}{\mu_2 v_2}}^{\frac{\mu_1 n_2}{\mu_2 n_1}} \frac{|E_t|^2}{|E_i|^2} \frac{\cos\theta_t}{\cos\theta_i}}_{T} = 1 \quad \text{(exercise)}$$

when projecting onto the z-axis we can think of the energy of the incoming wave being split on the reflected and transmitted ones – but this is not true in the x-direction

So far $n_1 < n_2$. If instead $n_2 < n_1$ then Snell's law gives $\theta_t = 90°$ when

$$\sin\theta_i = \frac{n_2}{n_1} \equiv \sin\theta_c$$

called critical angle (Problem 9.39)
For larger angles there is no transmitted wave

$$\cos\theta_t = \sqrt{1 - \frac{n_1^2}{n_2^2}\sin^2\theta_i} = i \underset{\nwarrow \text{ real}}{\xi}$$

and

$$\left.\frac{E_r}{E_i}\right|_{TM} = \frac{\mu_1 n_2 \cos\theta_i - i\mu_2 n_1 \xi}{+}$$

$$\left.\frac{E_r}{E_i}\right|_{TE} = \frac{\mu_2 n_1 \cos\theta_i - i\mu_1 n_2 \xi}{+}$$

so that in both cases $\left|\frac{E_r}{E_i}\right|^2 = 1$

[If we consider the evanescent wave we have

$$e^{i\vec{k}_t\cdot\vec{r}} = \underbrace{e^{i|\vec{k}_t| z\cos\theta_t}}_{e^{-|\vec{k}_t| z \xi}} e^{i|\vec{k}_t| x \sin\theta_t} =$$

[Snell's laws $|\vec{k}_i|\sin\theta_i = |\vec{k}_t|\sin\theta_t$, $n_1\sin\theta_i = n_2\sin\theta_t$]

$$= \underbrace{e^{-|\vec{k}_i|\frac{n_2}{n_1}\xi z}}_{\text{exp. damping}} \underbrace{e^{i|\vec{k}_i|\sin\theta_i x}}_{\text{propagation in } x\text{-dir.}}$$

writing the first factor as $e^{-\frac{z}{\delta}}$ we have

$$\delta = \frac{n_1}{|\vec{k}_i| n_2 \xi} \underset{\uparrow}{=} \frac{1}{|\vec{k}_i|\sqrt{\sin^2\theta_i - \frac{n_2^2}{n_1^2}}} \underset{\uparrow}{=} \frac{\lambda_i}{2\pi\sqrt{\sin^2\theta_i - \sin^2\theta_c}}$$

$$\xi = \left(\frac{n_1^2}{n_2^2}\sin^2\theta_i - 1\right)^{1/2} \qquad |\vec{k}_i| = \frac{2\pi}{\lambda_i}$$

finally the speed of propagation becomes

$$\frac{\omega}{|\vec{k}_i|\sin\theta_i} = \frac{c}{n_1 \sin\theta_i} = \frac{v_1}{\sin\theta_i} < v_2$$

$$\uparrow \sin\theta_i > \frac{n_2}{n_1}$$

note that there is no energy transported into medium 2 :

$$\langle \vec{S}_2 \rangle \cdot \hat{n} = \ldots = 0$$

Absorption and dispersion

In linear media the speed of propagation for a plane wave is

$$v = \frac{c}{n} = \frac{1}{\sqrt{\varepsilon\mu}} = \frac{1}{\sqrt{\varepsilon_0\mu_0}}\frac{1}{\sqrt{\varepsilon_r\mu_r}} \Rightarrow n = \sqrt{\varepsilon_r\mu_r}$$

So far we considered n real and constant.
Generalisations

$$n \text{ complex} \leftrightarrow \text{absorption}$$

$$n = n(\omega) \leftrightarrow \text{dispersion}$$

Simplest example of absorption – EM waves on a conductor.

Ohm's law: $\vec{J}_f = \sigma \vec{E}$ ← conductivity

What about ρ_f? Already argued that all free charges dissipate to surface of conductor. Explicitly, the continuity eqn gives

$$\frac{\partial}{\partial t}\rho_f = -\vec{\nabla}\cdot\vec{J} = -\sigma\vec{\nabla}\cdot\vec{E} = -\frac{\sigma}{\varepsilon}\rho_f \Rightarrow \rho_f = e^{-\frac{\sigma}{\varepsilon}t}\rho_f(0)$$

Maxwell's eqn's then become

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}\vec{B}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu\sigma\vec{E} + \mu\varepsilon\frac{\partial}{\partial t}\vec{E}$$

giving the following wave equations

$$\vec{\nabla}^2 \vec{E} = \mu\sigma \frac{\partial}{\partial t}\vec{E} + \mu\varepsilon \frac{\partial^2}{\partial t^2}\vec{E}$$

and the corresponding one for $\vec{B}$.

Plane wave solution

$$\vec{\tilde{E}}(\vec{r},t) = \hat{e}\,\tilde{E}\,e^{i(\vec{\tilde{k}}\cdot\vec{r}-\omega t)}$$

with (insert in wave-eqn to verify)

$$\vec{\tilde{k}}^{\,2} = \tilde{k}^2 = \mu\varepsilon\omega^2 + i\mu\sigma\omega = \mu\left(\varepsilon + \frac{i\sigma}{\omega}\right)\omega^2$$

complex wave-vector

can also define complex index of refraction

$$\tilde{k}^2 \equiv \frac{\tilde{n}^2}{c^2}\omega^2 \Rightarrow \tilde{n}^2 = \mu_r\varepsilon_r\left(1 + i\frac{\sigma}{\omega\varepsilon}\right)$$

writing $\tilde{k} = k + i\varkappa$ and $\tilde{n} = n + i\nu$ we also have

$k = n\frac{\omega}{c}$, $\varkappa = \nu\frac{\omega}{c}$ (implicitly this means that $\vec{\tilde{k}} = \vec{k} + i\varkappa\hat{k}$)

some gymnastics with complex numbers gives

$$n^2 = \frac{\varepsilon_r\mu_r}{2}\left(\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}+1\right) \qquad n^2-\nu^2 = \mu_r\varepsilon_r$$

$$\nu^2 = \frac{\varepsilon_r\mu_r}{2}\left(\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}-1\right) \qquad 2n\nu = \mu_r\varepsilon_r\frac{\sigma}{\omega\varepsilon}$$

Considering only <u>normal incidence</u> ($\vec{\tilde{k}} = (k+i\varkappa)\hat{z}$), we have

$$\vec{\tilde{E}}(\vec{r},t) = \hat{e}\,\tilde{E}\,\underbrace{e^{-\varkappa z}}_{\text{exponential damping, skin depth } d = \frac{1}{\varkappa}}\, e^{i(kz-\omega t)}$$

Faraday's law gives the corresponding $\vec{B}$-field

$$\tilde{\vec{B}}(\vec{r},t) = \hat{k}\times\hat{e}\ \frac{\tilde{k}}{\omega}\ \tilde{E}\ e^{-\varkappa z}\ \ e^{i(kz-\omega t)}$$

note: $\vec{B}$ and $\vec{E}$ have different phase

$$\tilde{k} = |\tilde{k}|\, e^{i\varphi} \quad , \quad \varphi = \arctan\frac{\varkappa}{k}$$

Boundary conditions:

$$\varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} = \sigma_f \qquad \vec{E}_1^{\parallel} - \vec{E}_2^{\parallel} = 0$$

$$B_1^{\perp} - B_2^{\perp} = 0 \qquad \frac{1}{\mu_1}\vec{B}_1^{\parallel} - \frac{1}{\mu_2}\vec{B}_2^{\parallel} = \vec{K}_f \times \hat{n}$$

normal incidence $\Rightarrow E_1^{\perp} = E_2^{\perp} = 0 \Rightarrow \sigma_f = 0$

ohmic conductor $(\vec{J}_f = \sigma\vec{E})$, $\vec{E}$ finite $\Rightarrow \vec{K}_f = 0$

∴ laws of refraction and reflection unchanged for normal incidence – only modifications $\tilde{n}_2$ and $\tilde{k}_t$ complex

(and $v, \varkappa \to \infty$)

note: for perfect conductor $(\sigma\to\infty)$ $n\to\infty$
$\Rightarrow$ total reflection (cf. mirror)

Alternatively we can define a complex $\tilde{\varepsilon}_r$

$$\tilde{\varepsilon}_r = \varepsilon_r + i\frac{\sigma}{\omega\varepsilon_0} \quad , \quad \vec{\nabla}^2\tilde{\vec{E}} = \mu\varepsilon_0\tilde{\varepsilon}_r\frac{\partial^2}{\partial t^2}\tilde{\vec{E}}$$

instead of having free currents

Complex $\tilde{\varepsilon}_r$ also arises more generally when describing dispersive materials

Dispersion:

Consider simple (classical) model for how the electrons in a material are affected by an electric field

$$\vec{F} = m_e \frac{d^2}{dt^2}\vec{r} = \underset{\substack{\uparrow \\ \text{spring const.}}}{-k\vec{r}} \underset{\substack{\uparrow \\ \text{damping}}}{-\alpha \frac{d}{dt}\vec{r}} \underset{\substack{\uparrow \\ \text{electric field}}}{-e\vec{E}}$$

rewrite as

$$\frac{d^2}{dt^2}\vec{r} + \frac{\alpha}{m_e}\frac{d}{dt}\vec{r} + \frac{k}{m_e}\vec{r} = -\frac{e}{m_e}\vec{E}$$

$\Rightarrow$ oscillatory motion with

eigenfrequency $\omega_0^2 = \frac{k}{m_e}$

damping const $\gamma = \frac{\alpha}{m_e}$

note: magnetic force neglected

assume harmonic time dep. $\tilde{\vec{E}} = \tilde{\vec{E}}(\vec{r})e^{-i\omega t}$

$$\Rightarrow \quad \tilde{\vec{r}} = -\frac{e}{m_e}\frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}\tilde{\vec{E}}$$

departure from equilibrium $\langle \tilde{\vec{r}} \rangle = 0$ gives rise to electric dipole moment

$$\tilde{\vec{p}} = -e\tilde{\vec{r}} = \frac{e^2}{m_e}\frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}\tilde{\vec{E}}$$

with N molecules per unit volume and f_j electrons with eigen frequency ω_j and damping γ_j per molecule we get the polarisation

$$\tilde{\vec{P}} = \underbrace{\frac{Ne^2}{m_e} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j}}_{\tilde{\chi}_e \varepsilon_0} \tilde{\vec{E}}$$

and the relative permittivity $\tilde{\varepsilon}_r = 1 + \tilde{\chi}_e$

Defining the plasma frequency

$$\omega_p^2 = \frac{Ne^2}{\varepsilon_0 m_e} \sum_j f_j \quad , \quad \sum_j f_j = Z$$

we finally get

$$\tilde{\varepsilon}_r = 1 + \frac{\omega_p^2}{Z} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j}$$

as before we have a complex index of refraction and setting $\mu_r = 1$ we have

$$\tilde{n}^2 = \tilde{\varepsilon}_r$$

with solutions $(\tilde{n} = n + i\nu)$

$$n = \frac{1}{\sqrt{2}} \sqrt{Re(\tilde{\varepsilon}_r) + |\tilde{\varepsilon}_r|}$$

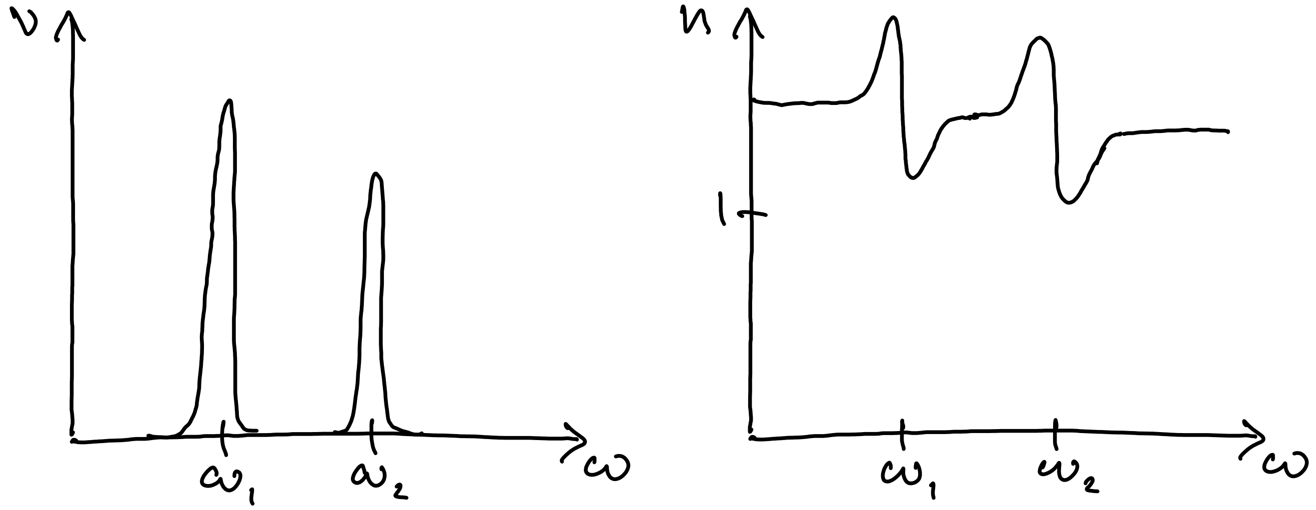
$$\nu = \frac{1}{2n} Im(\tilde{\varepsilon}_r)$$

Now

$$Re(\tilde{\varepsilon}_r) = 1 + \frac{\omega_p^2}{Z} \sum_j \frac{f_j(\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + \omega^2\gamma_j^2}$$

$$Im(\tilde{\varepsilon}_r) = \frac{\omega_p^2}{Z} \sum_j \frac{f_j \omega\gamma_j}{(\omega_j^2 - \omega^2)^2 + \omega^2\gamma_j^2}$$

so $\mathrm{Im}(\tilde{\varepsilon}_r) << \mathrm{Re}(\tilde{\varepsilon}_r) \Rightarrow \nu \approx 0$ unless $\omega \approx \omega_j$
i.e. at one of the eigen frequencies



(absorption lines in spectra from stars)

note that for $\omega >> \omega_j$ we have

$$\mathrm{Re}(\tilde{\varepsilon}_r) \approx 1 + \frac{\omega_p^2}{Z} \sum_j f_j \frac{-\omega^2}{\omega^4} = 1 - \frac{\omega_p^2}{\omega^2}$$

$$\mathrm{Im}(\tilde{\varepsilon}_r) \approx \frac{\omega_p^2}{Z} \sum_j f_j \frac{\omega \gamma_j}{\omega^4} \sim \frac{\omega_p^2}{\omega^3} \to 0$$

in other words $\varepsilon_r < 1$ and $n = \sqrt{\varepsilon_r} < 1$

$$\Rightarrow \quad v = \frac{c}{n} > c \quad !$$

information traveling faster than light?
No, plane wave cannot transmit info.

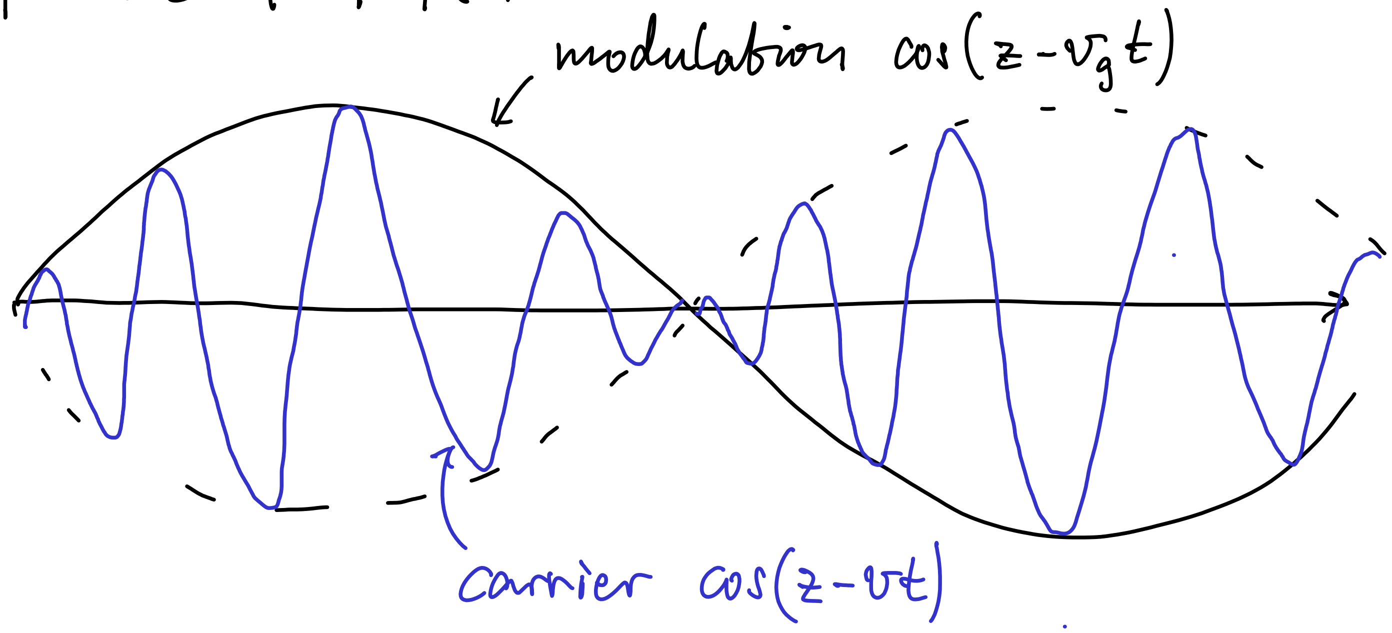
Consider signal consisting of two plane waves with different wave numbers and frequencies

$$\tilde{\vec{E}} = \hat{e}\tilde{E}\left[e^{i(k+\Delta k)z-(\omega+\Delta\omega)t} + e^{i(k-\Delta k)z-(\omega-\Delta\omega)t}\right]$$

$$= \hat{e}\tilde{E}\underbrace{\left[e^{i\Delta k z-\Delta\omega t} + e^{-i\Delta k z+\Delta\omega t}\right]}_{2\cos(\Delta k z-\Delta\omega t)} e^{i(kz-\omega t)}$$

signal with speed of prop. $v_g = \dfrac{\Delta\omega}{\Delta k}$

picture of $|\vec{E}|(z)$



More generally the group velocity is given by

$$v_g = \frac{d\omega}{dk}$$

in our case above we had

$$\varepsilon_r = n^2 = 1-\frac{\omega_p^2}{\omega^2} \Rightarrow \omega = \sqrt{\omega_p^2+c^2k^2}$$

$$\left[n^2 = c^2\frac{k^2}{\omega^2}\right]$$

so $$v_g = \frac{c^2 k}{\sqrt{\omega_p^2+c^2k^2}} = c^2\frac{k}{\omega} = nc < c$$

even more generally $k^2 = \frac{\omega^2}{v^2} = \frac{\omega^2}{c^2} \varepsilon_r$ gives

$$v_g = \left[\frac{dk}{d\omega} \right]^{-1} = \left[\frac{1}{v} \frac{k}{\omega} \frac{dk}{d\omega} \right]^{-1} =$$

$$= v \left[\frac{dk^2}{d\omega^2} \right]^{-1} = v \left[\frac{\varepsilon_r}{c^2} + \frac{\omega^2}{c^2} \frac{d\varepsilon_r}{d\omega} \right]^{-1}$$

$$= \frac{c}{\sqrt{\varepsilon_r} + \frac{\omega^2}{\sqrt{\varepsilon_r}} \frac{d\varepsilon_r}{d\omega^2}} = \frac{c}{\sqrt{\varepsilon_r} + \omega \frac{d\sqrt{\varepsilon_r}}{d\omega}}$$