Some answers to exam Aug 27, 2010

1a) $C = \frac{Q}{V_0} = 4\pi R \varepsilon_0$, $\quad W = 2\pi \varepsilon_0 V_0^2 R$

b) grounded sphere with radius R'; $V(R) = V_0$, $V(R') = 0$

$$\Rightarrow V = V_0 \frac{R}{r} \frac{R'-r}{R'-R} \quad , \vec{E} = \frac{V_0 R R'}{R'-R} \frac{\hat{r}}{r^2}$$

$$W' = 2\pi \varepsilon_0 V_0^2 R \frac{R'^2}{(R'-R)^2}$$

c) $C = \frac{Q}{V} = \frac{1.6 \cdot 10^{-19} \, As}{V} = 1.6 \cdot 10^{-19} \, F$

d) $R = \frac{Q}{4\pi \varepsilon_0 V_0} = \frac{1.6 \cdot 10^{-19} \, As \, m}{4\pi \cdot 8.85 \cdot 10^{-12} \, V F} \approx 1 \cdot 10^{-9} \, m$

2a) $\dot{\vec{B}} = -i\omega \vec{B}$

$$\vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot \left(\frac{i}{\omega} \dot{\vec{B}} \right) = -\vec{\nabla} \cdot \left(\frac{i}{\omega} \vec{\nabla} \times \vec{E} \right) = 0$$

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{v}) = 0$$

b) ME:
$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\varepsilon_0} \rho \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t} \vec{B}$$
$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \vec{E} + \mu_0 \vec{J}$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} (\vec{\nabla} \cdot \vec{E}) - \vec{\nabla}^2 \vec{E} = \vec{\nabla} \times \left(-\frac{\partial}{\partial t} \vec{B} \right) =$$
$$= -\frac{\partial}{\partial t} (\vec{\nabla} \times \vec{B}) = -\mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \vec{E} - \mu_0 \frac{\partial}{\partial t} \vec{J}$$

$$\Rightarrow \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \vec{E} - \vec{\nabla}^2 \vec{E} + \vec{\nabla} \left(\frac{1}{\varepsilon_0} \rho \right) = -\mu_0 \frac{\partial}{\partial t} \vec{J}$$

$$-\omega^2 \varepsilon_0 \mu_0 \vec{E} + k_x^2 \vec{E} - e^{i(k_x x - \omega t)} \frac{\partial^2}{\partial z^2} \vec{E}(z) + \vec{\nabla}\left(\frac{1}{\varepsilon_0} \rho\right) =$$

$$= i\omega\mu_0 \vec{J}$$

$$\vec{\nabla} \rho = \left(\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}\right) \rho(z)\, e^{i(k_x x - \omega t)} =$$

$$= \hat{x}\,(ik_x)\,\rho(z)\, e^{i(k_x x - \omega t)} + \hat{z} \frac{\partial}{\partial z} \rho(z)\, e^{i(k_x x - \omega t)}$$

3 a) $\vec{K} \simeq 10^6 \frac{A}{m}$

b) sign ?

c) $W = \int \vec{F} \cdot d\vec{l}$, $\frac{dW}{dt} = \int (\vec{E} \cdot \vec{J})\, d^3\vec{r}$ ✓

$\vec{J} = \sigma \vec{E}$, $\vec{E}^2 = \sigma v^2 B_x^2$

d) $|\vec{F}| = \frac{1}{v} \frac{dW}{dt} = \sigma v B_0^2 \int \sin^2 \frac{\pi z}{l}\, d^3\vec{r}'$

e) $\langle \vec{J} \rangle = \sigma v B_0 \frac{1}{2} = 3.5 \cdot 10^7 \cdot 20 \cdot 0.6 \frac{1}{2} \frac{A}{m^2}$

f) $|\vec{F}| \propto v$

4. n/a

Some answers to exam June 1, 2011

1a) $W_{12} = -\frac{e^2}{4\pi\varepsilon_0(r_+ + r_-)}$

b) $W = W_{12} + W_{13} + W_{23} + W_{14} + W_{24} + W_{34} =$

$= -\frac{e^2}{4\pi\varepsilon_0(r_+ + r_-)}\left(1 - \frac{1}{2} + 1 + \frac{1}{3} - \frac{1}{2} + 1\right) = \frac{7}{3} W_{12}$

c) $E_b = 1.6 \cdot 10^{-9} \cdot 9.0 \cdot 10^9 \frac{1}{2.8} 10^{10} \frac{\text{Cm}}{\text{Fm}} e \approx 5\,\text{eV}$

2a) $g_{TE} = \frac{n_2 \cos\theta_2}{n_1 \cos\theta_1} \qquad g_{TM} = \frac{n_2 \cos\theta_1}{n_1 \cos\theta_2}$

$\rho_1 = \frac{(1-g)^2}{(1+g)^2}$

$\rho_2 = \frac{(1-g'_{TE})^2}{(1+g'_{TE})^2} = \frac{(n_2\cos\theta_2 - n_1\cos\theta_1)^2}{(n_2\cos\theta_2 + n_1\cos\theta_1)^2} = \frac{(g_{TE}-1)^2}{(g_{TE}+1)^2}$

b) $\rho_{tot} = \rho_1 + (1-\rho_1)\rho_2(1-\rho_2) +$

$+ (1-\rho_1)\rho_2\rho_2\rho_2(1-\rho_2) + (1-\rho_1)(\rho_2)^5(1-\rho_2) + \ldots$

$= [\rho_1 = \rho_2 = \rho]$

$= \rho\left[1 + (1-\rho)^2 \underbrace{\left(1 + \rho^2 + \rho^4 + \ldots\right)}_{\frac{1}{1-\rho^2}}\right] = \frac{2\rho}{1+\rho}$

3a) $\frac{dW}{dt} = \int(\vec{J}\cdot\vec{E})\,d^3\vec{r} = [\vec{J} = \sigma\vec{E}] = \int \sigma \vec{E}^2\, d^3\vec{r}$

b) $\vec{B} = \mu_0 \vec{M}\, \Theta(h_0 - vt) \Rightarrow -\frac{\partial}{\partial t}\vec{B} = \mu_0 v M \cdot C$

$$\frac{dW}{dt} = \vec{F} \cdot \vec{v}$$

$$\Rightarrow |\vec{F}| = \frac{1}{v} \sigma v^2 M^2 \mu_0^2 c^2 \pi R b L$$

$$= c' \mu_0^2 M^2 \sigma b R^2 v$$

4 a) ME in LG: $\partial^2 A^\mu = \mu_0 J^\mu$

$$\partial^2 = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right), \quad A^\mu = \left(\frac{1}{c}V, \vec{A}\right), \quad J^\mu = (c\rho, \vec{J})$$

$$\Rightarrow \left(\frac{1}{c^2}(-i\omega)^2 - (ik_z)^2 - \partial_x^2 - \partial_y^2\right) A^\mu(x,y) = \mu_0 J^\mu(x,y)$$

LG: $\frac{1}{c}\frac{\partial}{\partial t} A^0 + \vec{\nabla} \cdot \vec{A} = 0$

$$\Rightarrow -\frac{i\omega}{c} A^0(x,y) + ik_z A^3(x,y) + \partial_x A^1(x,y) + \partial_y A^2(x,y) = 0$$

b) wave along wave-guide

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2}\right) A^\mu(x,y) = 0$$

$$\Rightarrow (\partial_x^2 + \partial_y^2) A^\mu(x,y) + \mu_0 J^\mu(x,y) = 0$$

c) $\vec{E} = -\vec{\nabla} V - \frac{\partial}{\partial t}\vec{A}$

$$\Rightarrow E_z = -ik_z c A^0(x,y) e^{i(k_z z - \omega t)} + i\omega A^3(x,y) e^{i(k_z z - \omega t)}$$

$$= 0$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial}{\partial t}\vec{E}$$

Some answers to exam May 25, 2012:

1a) $W = V(\vec{r}_1)q - V(\vec{r}_2)q = q\left[V(\vec{r}_1) - V(\vec{r}_2)\right]$

$= qh \dfrac{V(\vec{r}_2+\vec{h}) - V(\vec{r}_2)}{h}$

$h \to 0, \quad q \to \infty, \quad qh = p = \mathrm{const}$

$\Rightarrow W = \vec{p} \cdot \vec{\nabla} V = -\vec{p} \cdot \vec{E}$

b) $\vec{E} = -\dfrac{q(\vec{r}_1 - \vec{r}_2)}{4\pi\varepsilon_0 |\vec{r}_1 - \vec{r}_2|^3}$

$$\left.\begin{aligned} \Rightarrow W &= \frac{q\,\vec{p}\cdot(\vec{r}-\vec{r}_2)}{4\pi\varepsilon_0|\vec{r}-\vec{r}_2|^3} \\ V_{ed} &= \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0|\vec{r}|^3} \end{aligned}\right\} \quad W = qV$$

2a) $\langle I \rangle = \dfrac{1}{2\mu_0 v} |\tilde{E}|^2$

$v = 3 \cdot 10^8\, \mathrm{m/s}$

$|\tilde{E}| = \sqrt{2\mu_0 v \langle I \rangle} \approx 3 \cdot 10^{-1}\, \dfrac{V}{m}$

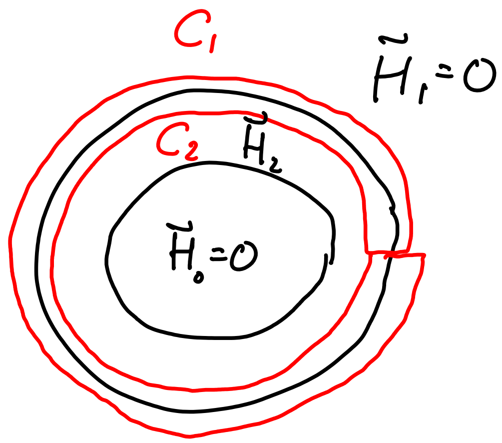
$|\tilde{B}| \approx 10^{-9}\, \dfrac{Vs}{m^2}$

b) $|\tilde{E}| \approx 100\, GV \dfrac{1}{m} = 10^{11}\, \dfrac{V}{m}$

$\Rightarrow \langle I \rangle \approx 10^{19}\, \dfrac{W}{m^2} \iff 3 \cdot 10^{21}\, \mathrm{lux}$

3a) Cyl. coord $\quad \vec{B} = -\dfrac{\mu_r \mu_0 N I}{2\pi s}\,\hat{\varphi}$

b) $\vec{H}_2 = \dfrac{1}{\mu}\vec{B} = -\dfrac{NI}{2\pi s}\,\hat{\varphi}$



$$\int_{C_1} \vec{H}_1 \cdot d\vec{l} + \int_{C_2} \vec{H}_2 \cdot d\vec{l} = NI$$

c) $L = \dfrac{\Phi}{I}$

$$\Phi_{1\text{-turn}} = \int \vec{B}\cdot d\vec{S} = \frac{\mu N I}{2\pi}\, a \int_{R-b}^{R} \frac{1}{s}\,ds = \frac{\mu N I}{2\pi}\, a \ln\frac{R}{R-b}$$

$$\Phi_{tot} = N\,\Phi_{1\text{-turn}}$$

$$\Rightarrow L = \frac{\mu N^2}{2\pi}\, a \ln\frac{R}{R-b}$$

4a) $\tilde{\vec{E}}(\vec{r},t) = E\hat{e}\,\frac{1}{2}\left(e^{i(\vec{k}\cdot\vec{r})} + e^{-i(\vec{k}\cdot\vec{r})}\right)e^{-i\omega t}$

b) $\vec{\nabla}\times\vec{E} = -\dfrac{\partial}{\partial t}\vec{B} = i\omega\vec{B}$

$$\tilde{\vec{B}} = \frac{1}{i\omega} E \frac{1}{2}\left(i\vec{k}\times\hat{e}\, e^{i(\vec{k}\cdot\vec{r})} - i\vec{k}\times\hat{e}\, e^{-i(\vec{k}\cdot\vec{r})}\right)e^{-i\omega t}$$

$$= E\,\frac{\vec{k}\times\hat{e}}{\omega}\; i\sin(\vec{k}\cdot\vec{r})\, e^{-i\omega t}$$

$$\vec{B} = E\,\frac{1}{\omega}(\vec{k}\times\hat{e})\sin(\vec{k}\cdot\vec{r})\sin(\omega t)$$

c) LG: $\dfrac{1}{c^2}\dfrac{\partial}{\partial t}V + \vec{\nabla}\cdot\vec{A} = 0$

$$\vec{A} = -E\,\hat{e}\,\frac{1}{\omega}\cos(\vec{k}\cdot\vec{r})\sin(\omega t), \quad V = 0$$

Some answers to exam June 17, 2011

1a, use Gauss's law

b, use $\vec{E} = -\vec{\nabla} V$

c, $W_e = \frac{1}{2}\varepsilon_0 \int \vec{E}^2 d^3\vec{r} =$

$$= \frac{1}{2}\frac{Q^2}{4\pi\varepsilon_0}\left[\int_0^R \frac{r^4}{R^6}dr + \int_R^\infty \frac{1}{r^2}dr\right]$$

d, $W_{Ar} = \frac{3\cdot(18)^2 e^2}{20\pi\varepsilon_0 R}$, $W_{Cl} = \frac{3\cdot(17)^2 e^2}{20\pi\varepsilon_0 R}$

$$W_{Ar} - W_{Cl} = 7.26\,\text{MeV} = \frac{3e}{20\pi\varepsilon_0}\frac{(18)^2-(17)^2}{R}e$$

$$R = 0.86\cdot 10^{-9}\,\text{Vm}\,\frac{18^2-17^2}{7.26\cdot 10^6\,\text{V}} \approx 4\cdot 10^{-15}\,\text{m}$$

2a, $n = 1.5$, $\mu_r = 1 \Rightarrow \varepsilon_r = n^2 = 2.25$

$\vec{P} = \chi_e\varepsilon_0\vec{E}$, $\varepsilon_r = 1+\chi_e \Rightarrow \alpha = \chi_e = 1.25$

b, $I_i = \frac{1}{2\mu_0 v_i}|\tilde{E}_i|^2$, $v_i = c$

$I_t = \frac{1}{2\mu_0 v_t}|\tilde{E}_t|^2$, $v_t = \frac{c}{n}$ $\qquad \frac{v_i}{v_t} = n$

$$\frac{|\tilde{E}_t|}{|\tilde{E}_i|} = \frac{2}{1+n}$$

$$\Rightarrow \frac{I_t}{I_i} = \frac{v_i}{v_t}\left(\frac{2}{1+n}\right)^2 = \frac{4n}{(1+n)^2} = \frac{6}{6.25}$$

c, $|\tilde{E}_t| = \frac{2}{2.5}|\tilde{E}_i| \approx 7.0\,\frac{\text{V}}{\text{m}}$

3. n/a (optics)

4 a) $\tilde{\vec{A}} = \frac{\mu_0}{4\pi}(\hat{r} \times \vec{m})\left(\frac{ik}{r} - \frac{1}{r^2}\right)e^{i(kr-\omega t)}$

$\sim \frac{1}{r}$

b) $\tilde{\vec{B}} = \vec{\nabla} \times \tilde{\vec{A}}$

$\vec{\nabla} \times \left(\frac{e^{ikr}}{r}\vec{v}\right) = \left(ik\frac{e^{ikr}}{r} - \frac{e^{ikr}}{r^2}\right)\hat{r} \times \vec{v}$

const

$\Rightarrow \tilde{\vec{B}} = \frac{\mu_0}{4\pi}\hat{r} \times (\hat{r} \times \vec{m})\frac{(ik)^2}{r}e^{i(kr-\omega t)} =$

$[kr \gg 1] \quad = \frac{\mu_0 k^2}{4\pi r}(\hat{r} \times \vec{m}) \times \hat{r}\, e^{i(kr-\omega t)}$

c) $\mu_0 \varepsilon_0 \frac{\partial}{\partial t}\vec{E} = \frac{-i\omega}{c^2}\vec{E} = \vec{\nabla} \times \vec{B}$

$\Rightarrow \tilde{\vec{E}} = \frac{ic^2}{\omega}\frac{\mu_0 k^2}{4\pi r}\, ik\,\hat{r} \times [(\hat{r} \times \vec{m}) \times \hat{r}]\, e^{i(kr-\omega t)}$

$[kr \gg 1]$

$= \frac{\mu_0 c k^2}{4\pi r}(\vec{m} \times \hat{r})\, e^{i(kr-\omega t)}$