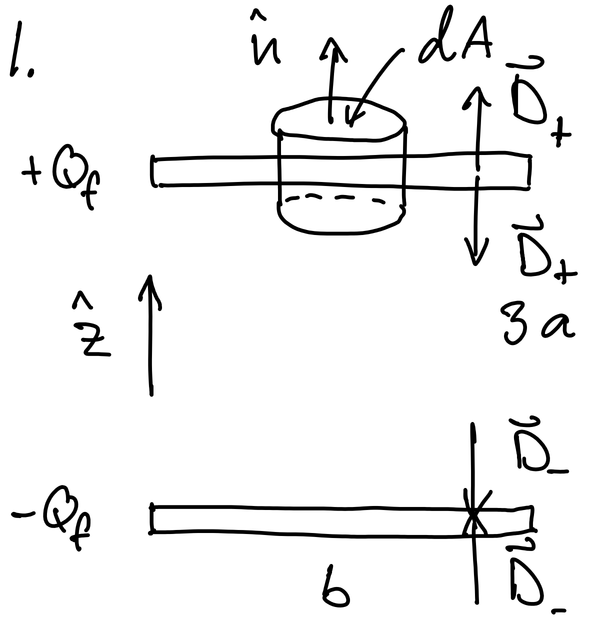
Electromagnetism exam June 3, 2016

Sketches for solutions

1.



$a \ll b$ no edge effects

a) Gauss law on upper plate

$$\int \vec{D}_+ \cdot d\vec{A} = Q_{f,enc}$$

symmetry gives $\vec{D}_+ = D_+ \hat{z}$ above and $\vec{D}_+ = -D_+ \hat{z}$ below the plate

surface charge $\sigma_f = \frac{Q_f}{b^2}$

$$\Rightarrow \; 2 D_+ \, dA = \frac{Q_f}{b^2} \, dA \quad , \quad D_+ = \frac{\sigma_f}{2}$$

For the lower plate we get in the same way that $D_- = \sigma_f / 2$

Adding the two solutions we find

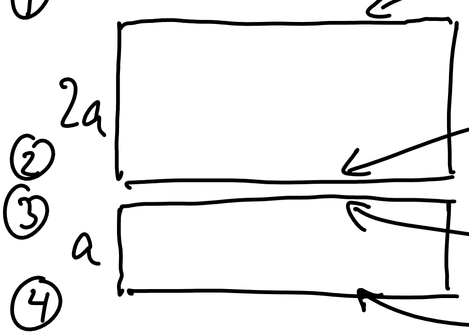
$$\vec{D} = \begin{cases} -\sigma_f \hat{z} & \text{between the plates} \\ 0 & \text{outside the plates} \end{cases}$$

b) by definition we know that $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$
for a linear medium $\vec{P} = \chi_e \varepsilon_0 \vec{E}$

$$\Rightarrow \vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 (1 + \chi_e) \vec{E} = \varepsilon_0 \varepsilon_r \vec{E}$$

For the upper slab we get: $\vec{E} = \frac{1}{2\varepsilon_0} \vec{D}$, $\vec{P} = \frac{1}{2} \vec{D}$
and for the lower one: $\vec{E} = \frac{1}{3\varepsilon_0} \vec{D}$, $\vec{P} = \frac{2}{3} \vec{D}$

c) the bound surface charge is given by $\sigma_b = \hat{n} \cdot \vec{P}$



$\sigma_b = \hat{z} \cdot \left(-\frac{1}{2}\sigma_f \hat{z}\right) = -\frac{1}{2}\sigma_f$, $\sigma = \frac{1}{2}\sigma_f$

$\sigma_b = -\hat{z} \cdot \left(-\frac{1}{2}\sigma_f \hat{z}\right) = \frac{1}{2}\sigma_f$, $\sigma = \frac{1}{2}\sigma_f$

$\sigma_b = \hat{z} \cdot \left(-\frac{2}{3}\sigma_f \hat{z}\right) = -\frac{2}{3}\sigma_f$, $\sigma = -\frac{2}{3}\sigma_f$

$\sigma_b = -\hat{z} \cdot \left(-\frac{2}{3}\sigma_f \hat{z}\right) = \frac{2}{3}\sigma_f$, $\sigma = -\frac{1}{3}\sigma_f$

the discontinuity in the $\vec{E}$-field is given by

$$\left(\vec{E}_{above} - \vec{E}_{below}\right) \cdot \hat{z} = \frac{1}{\varepsilon_0} \sigma = \frac{1}{\varepsilon_0} (\sigma_f + \sigma_b)$$

starting from the upper plate we have

1 $\quad 0 - \left(-\frac{1}{2\varepsilon_0}\sigma_f\right) = \frac{1}{2\varepsilon_0}\sigma_f$ ok

2 $\quad -\frac{1}{2\varepsilon_0}\sigma_f - \left(-\frac{1}{\varepsilon_0}\sigma_f\right) = \frac{1}{2\varepsilon_0}\sigma_f$ ok

3 $\quad -\frac{1}{\varepsilon_0}\sigma_f - \left(-\frac{1}{3\varepsilon_0}\sigma_f\right) = -\frac{2}{3\varepsilon_0}\sigma_f$ ok

4 $\quad -\frac{1}{3\varepsilon_0}\sigma_f - 0 = -\frac{1}{3\varepsilon_0}\sigma_f$ ok

2. Potential known on spherical shell

$$V(r=R,\theta,\varphi) = k\cos\theta$$

a) no charges $\Rightarrow$ $\vec{\nabla}^2 V = 0$ except on shell

Spherical symmetry $\Rightarrow$ we can write general solution as

$$V(r,\theta) = \sum_{\ell} \left(A_\ell r^\ell + B_\ell \frac{1}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

coefficients A_ℓ, B_ℓ fixed by bound. cond.

inside: V regular at $r=0$ $\Rightarrow$ $B_\ell = 0$

$P_1(\cos\theta) = \cos\theta$, P_ℓ orthogonal when integrating between $\cos\theta=-1$ and $\cos\theta=1$

$\Rightarrow$ only $A_1 \neq 0$

Bound. cond $\Rightarrow$ $A_1 R\cos\theta = k\cos\theta \Rightarrow A_1 = \frac{k}{R}$

$$\Rightarrow \quad V_{in} = \frac{k}{R} r\cos\theta$$

outside: V vanishes at infinity $\Rightarrow$ $A_\ell = 0$

$P_1(\cos\theta) = \cos\theta$ $\Rightarrow$ only $B_1 \neq 0$

Bound cond $\Rightarrow$ $B_1 \frac{1}{R^2}\cos\theta = k\cos\theta \Rightarrow B_1 = kR^2$

$$\Rightarrow \quad V_{out} = kR^2 \frac{1}{r^2}\cos\theta$$

b) By definition $\vec{E} = -\vec{\nabla} V$ so

$$\vec{E}_{in} = -\hat{r}\frac{k}{R}\cos\theta + \hat{\theta}\frac{k}{R}\sin\theta$$

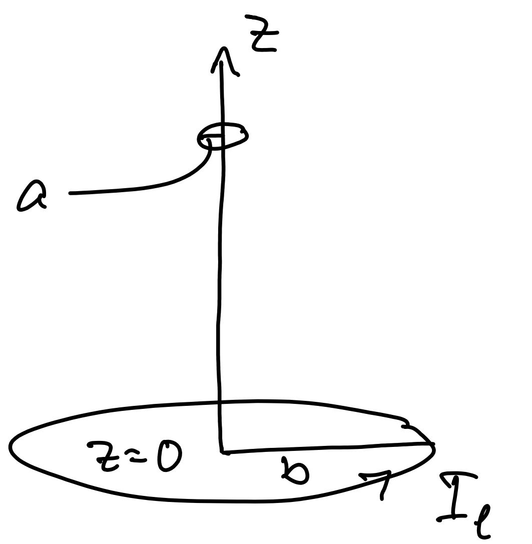
$$\vec{E}_{out} = +\hat{r}\,\frac{2kR^3}{r^3}\cos\theta + \hat{\theta}\,\frac{kR^3}{r^3}\sin\theta$$

c) Boundary conditions

$$\left(\vec{E}_{out} - \vec{E}_{in}\right)\Big|_{r=R}\cdot\hat{r} = \frac{1}{\varepsilon_0}\sigma$$

$$\Rightarrow \sigma = \varepsilon_0\, 3\,\frac{k}{R}\cos\theta$$

3.



$a \ll b$

a) Biot-Savart: $B(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,')\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\, d^3\vec{r}\,'$

Cylindrical coord: $\vec{r} = z\hat{z}$, $\vec{r}\,' = b\hat{s}$, $|\vec{r}-\vec{r}\,'| = \sqrt{z^2+b^2}$

$(\hat{x}\cos\varphi + \hat{y}\sin\varphi$

The current: $\vec{J}(\vec{r}\,') = I_\ell\hat{\varphi}\,\delta(s-b)\,\delta(z)$

$-\hat{x}\sin\varphi + \hat{y}\cos\varphi$

Cross product:

$$\vec{J}(\vec{r}\,')\times(\vec{r}-\vec{r}\,') = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\sin\varphi & \cos\varphi & 0 \\ -b\cos\varphi & -b\sin\varphi & z \end{vmatrix} =$$

$$= \hat{x}\, z\cos\varphi + \hat{y}\, z\sin\varphi + \hat{z}\underbrace{\left(b\sin^2\varphi + b\cos^2\varphi\right)}_{b}$$

The field on the z-axis is then given by

$$\vec{B}(\vec{r}) = \frac{\mu_0 I_e}{4\pi}\int \frac{(\hat{x}\, z\cos\varphi + \hat{y}\, z\sin\varphi + \hat{z}\, b)}{(b^2+z^2)^{3/2}}\, b\, d\varphi$$

$$= \frac{\mu_0 I_e}{2}\frac{b^2}{(b^2+z^2)^{3/2}}\,\hat{z}$$

b) Faradays law, $\vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}\vec{B}$

on integral form $\oint \vec{E}\cdot d\vec{l} = \int -\frac{\partial}{\partial t}\vec{B}\cdot d\vec{S}$

$a \ll b \Rightarrow \vec{B}$ and $\frac{\partial}{\partial t}\vec{B}$ const over small loop

$$\Rightarrow \int -\frac{\partial}{\partial t}\vec{B}\cdot d\vec{S} = \pi a^2\left(-\frac{\mu_0 \dot{I}_e}{2}\frac{b^2}{(b^2+z^2)^{3/2}}\right)$$

$\frac{\partial}{\partial t}\vec{B} = C\hat{z}$ with $C>0$ and the right-hand rule $\Rightarrow \vec{E} = -E\hat{\varphi}$

so $\int \vec{E}\cdot d\vec{l} = -2\pi a E$ and

$$\vec{E} = -\frac{\mu_0 \dot{I}_e}{4}\frac{a b^2}{(b^2+z^2)^{3/2}}\,\hat{\varphi}$$

Ohm's law then gives $\vec{J}_s = \sigma\vec{E}$

The current goes in the $-\hat{\varphi}$-direction, which is in agreement with Lenz law.

4. Point like dipole $\vec{\tilde{p}} = \tilde{p}_0 e^{-i\omega t}\hat{z}$ at $\vec{r}' = 0$.

Retarded solution

$$\tilde{A}^\mu = \frac{\mu_0}{4\pi}\int \frac{\tilde{J}^\mu\left(\vec{r}', t - \frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|}\, d^3\vec{r}'$$

The three-vector part is then given by

$$\tilde{\vec{A}}_{ed} = \frac{\mu_0}{4\pi}\frac{1}{r}\int \vec{J}^{\mu}\left(\vec{r}', t-\frac{|\vec{r}-\vec{r}'|}{c}\right) d^3\vec{r}' = [\text{use hint}] =$$

$$= \frac{\mu_0}{4\pi}\frac{1}{r}\frac{\partial}{\partial t}\tilde{\vec{p}}\left(t-\frac{r}{c}\right) = -i\omega\frac{\mu_0}{4\pi}\frac{1}{r}\tilde{p}_0 e^{-i\omega\left(t-\frac{r}{c}\right)}\hat{z}$$

$$\left[\frac{\omega}{c}=k\right] = -i\omega\frac{\mu_0}{4\pi}\frac{1}{r}e^{i(kr-\omega t)}\tilde{p}_0\hat{z}$$

b) by definition $\tilde{\vec{B}} = \vec{\nabla}\times\tilde{\vec{A}}$

in the radiation zone $kr \gg 1$ and

$$\vec{\nabla}\times\left(\frac{e^{ikr}}{r}\hat{z}\right) = i\vec{k}\times\hat{z}\,\frac{e^{ikr}}{r}$$

$$\Rightarrow \tilde{\vec{B}} = \frac{\mu_0}{4\pi}\frac{\omega k}{r}e^{i(kr-\omega t)}\tilde{p}_0\underbrace{\hat{r}\times\hat{z}}_{-\hat{\varphi}\sin\theta}$$

Ampere's law without sources and with harmonic time dependence

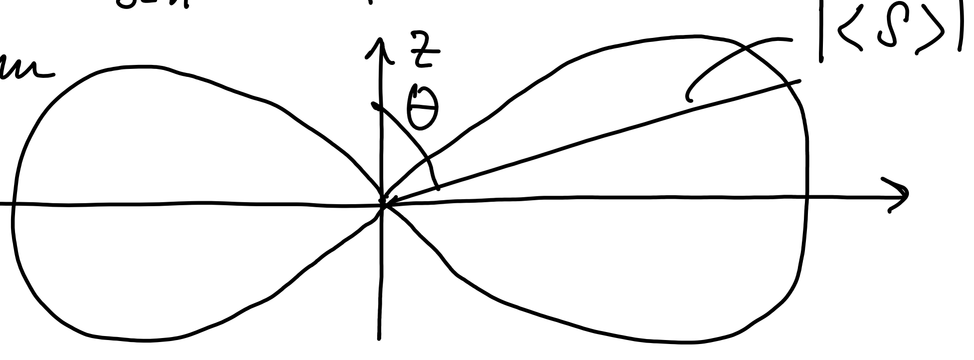
$$\vec{\nabla}\times\tilde{\vec{B}} = \mu_0\varepsilon_0\frac{\partial}{\partial t}\tilde{\vec{E}} = \frac{1}{c^2}(-i\omega)\tilde{\vec{E}}$$

$$\Rightarrow \tilde{\vec{E}} = -\frac{c^2}{\omega}\frac{\mu_0}{4\pi}\frac{\omega k^2}{r}e^{i(kr-\omega t)}\tilde{p}_0\underbrace{\hat{r}\times(\hat{r}\times\hat{z})}_{\hat{\theta}\sin\theta}$$

c) Poynting vector

$$\langle\vec{S}\rangle = \frac{1}{2\mu_0}\tilde{\vec{E}}\times\tilde{\vec{B}}^{*} = \frac{1}{32\pi^2}\mu_0 c^2\frac{\omega k^3}{r^2}|p_0|^2\sin^2\theta\,\hat{r}$$

Antenna diagram



5. a) Maxwell's equations

$$\vec\nabla\cdot\vec E = \frac{1}{\varepsilon_0}\rho \qquad \vec\nabla\times\vec E + \frac{\partial}{\partial t}\vec B = 0$$

$$\vec\nabla\cdot\vec B = 0 \qquad \vec\nabla\times\vec B - \mu_0\varepsilon_0\frac{\partial}{\partial t}\vec E = \mu_0\vec J$$

$$\vec\nabla\cdot\vec B = 0 \quad \Rightarrow \quad \vec B = \vec\nabla\times\vec A$$

$$\vec\nabla\times\vec E + \frac{\partial}{\partial t}(\vec\nabla\times\vec A) = 0 \quad \Rightarrow \quad \vec\nabla\times\left(\vec E + \frac{\partial}{\partial t}\vec A\right) = 0$$

$$\Rightarrow \quad \vec E + \frac{\partial}{\partial t}\vec A = -\vec\nabla V \quad , \quad \vec E = -\vec\nabla V - \frac{\partial}{\partial t}\vec A$$

b) Magnetic field, $\vec B = \begin{cases} B\hat z & , \ s < s_0 \\ 0 & , \ s > s_0 \end{cases}$

$$\vec E = 0, \ V = 0 \ \Rightarrow \ \frac{\partial}{\partial t}\vec A = 0$$

Curl in cylindrical coordinates

$$\vec\nabla\times\vec a = \hat s\underbrace{\left[\frac{1}{s}\frac{\partial}{\partial\varphi}a_z - \frac{\partial}{\partial z}a_\varphi\right]}_{=0} + \hat\varphi\underbrace{\left[\frac{\partial}{\partial z}a_s - \frac{\partial}{\partial s}a_z\right]}_{=0} + \hat z\underbrace{\frac{1}{s}\left[\frac{\partial}{\partial s}(sa_\varphi) - \frac{\partial}{\partial\varphi}a_s\right]}_{\begin{cases} B & , \ s<s_0 \\ 0 & , \ s>s_0\end{cases}}$$

$$\text{Solution:} \quad A_\varphi = \begin{cases} \dfrac{Bs}{2} & , \ s < s_0 \\ \dfrac{Bs_0^2}{2s} & , \ s > s_0 \end{cases}$$

c) The phase-shift becomes

$$\Delta\varphi = \pm\frac{q}{\hbar}\oint\vec A\cdot d\vec l = \pm\frac{q}{\hbar}\frac{Bs_0^2}{2}\underbrace{\int\frac{\hat\varphi}{s}\cdot\hat\varphi\, s\, d\varphi}_{2\pi}$$

or alternatively

$$\Delta\varphi = \pm\frac{q}{\hbar}\oint\vec A\cdot d\vec l = \pm\frac{q}{\hbar}\int(\vec\nabla\times\vec A)\cdot d\vec S =$$

$$= \pm\frac{q}{\hbar}\int\vec B\cdot d\vec S = \pm\frac{q}{\hbar}\pi s_0^2 B$$