Electromagnetism - June 5, 2015

1a, Symmetry $\Rightarrow$ D-field from Gauss's law

$$\vec{\nabla}\cdot\vec{D} = \rho_f \Rightarrow \int \vec{D}\cdot d\vec{S} = Q_{f,enc} \quad , \quad \vec{D} = \frac{Q_{f,enc}}{4\pi r^2}\hat{r}$$

charge uniformly distributed within $r<a$

$$\Rightarrow \rho_f = \frac{3\cdot Q_f}{4\pi a^3} \quad , \quad Q_{f,enc} = \frac{4\pi r^3}{3}\rho_f = Q_f\frac{r^3}{a^3}$$

no charge in $a<r<b$

$$\rho_f = 0 \quad , \quad Q_{f,enc} = Q_f$$

spherical shell at $r=b$

$$\sigma_f = -\frac{Q_f}{4\pi b^2} \qquad \Rightarrow \qquad Q_{f,enc} = 0 \quad r>b$$

Putting everything together gives

$$\vec{D} = \begin{cases} \dfrac{Q_f\, r}{4\pi a^3}\hat{r} & , \ r<a \\ \dfrac{Q_f}{4\pi r^2}\hat{r} & , \ a<r<b \\ 0 & , \ r>b \end{cases}$$

b, Linear medium $\vec{D} = \varepsilon\vec{E} = \varepsilon_0\varepsilon_r\vec{E} = \varepsilon_0\vec{E} + \vec{P}$

total charge density $\rho = \varepsilon_0\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_r}\rho_f$

$$\Rightarrow \vec{E} = \begin{cases} \dfrac{Q_f\, r}{4\pi\varepsilon_0\varepsilon_r a^3}\hat{r} \\ \dfrac{Q_f}{4\pi\varepsilon_0 r^2}\hat{r} \\ 0 \end{cases} , \quad \rho = \begin{cases} \dfrac{3Q_f}{4\pi\varepsilon_r a^3} & r<a \\ 0 & a<r<b \\ 0 & r>b \end{cases}$$

Polarisation field: $\vec{P} = \vec{D} - \varepsilon_0 \vec{E} = \varepsilon_0(\varepsilon_r - 1)\vec{E} = \frac{\varepsilon_r - 1}{\varepsilon_r}\vec{D}$

bound charge density $\rho_b = -\vec{\nabla}\cdot\vec{P} = \rho - \rho_f = \frac{1-\varepsilon_r}{\varepsilon_r}\rho_f$

$$\vec{P} = \begin{cases} \frac{\varepsilon_r - 1}{\varepsilon_r}\frac{Q_f r}{4\pi a^3}\hat{r} \\ 0 \\ 0 \end{cases}, \quad \rho_b = \begin{cases} \frac{1-\varepsilon_r}{\varepsilon_r}\frac{3Q_f}{4\pi a^3} & r < a \\ 0 & a < r < b \\ 0 & r > b \end{cases}$$

c) bound surface charge

$$\sigma_b = \vec{P}\cdot\hat{r}\Big|_{r=a} = \frac{\varepsilon_r - 1}{\varepsilon_r}\frac{Q_f}{4\pi a^2}$$

total charge inside $r = a - \delta$:

$$Q = \int^{r<a} \rho \, d^3\vec{r} = \int^{r<a} (\rho_f + \rho_b)\, d^3\vec{r} = \frac{Q_f}{\varepsilon_r} = Q_f - \frac{\varepsilon_r - 1}{\varepsilon_r}Q_f$$

total charge inside $r = a + \delta$:

$$Q = \frac{Q_f}{\varepsilon_r} + 4\pi a^2 \sigma_b = Q_f\left(\frac{1}{\varepsilon_r} + \frac{\varepsilon_r - 1}{\varepsilon_r}\right) = Q_f$$

boundary conditions for $\vec{E}$-field

$$(\vec{E}_{out} - \vec{E}_{in})\cdot\hat{r} = \frac{1}{\varepsilon_0}\sigma = \frac{1}{\varepsilon_0}\sigma_b$$

$$\frac{Q_f}{4\pi\varepsilon_0 a^2} - \frac{Q_f}{4\pi\varepsilon_0\varepsilon_r a^2} = \frac{1}{\varepsilon_0}\frac{\varepsilon_r - 1}{\varepsilon_r}\frac{Q_f}{4\pi a^2} \qquad \square$$

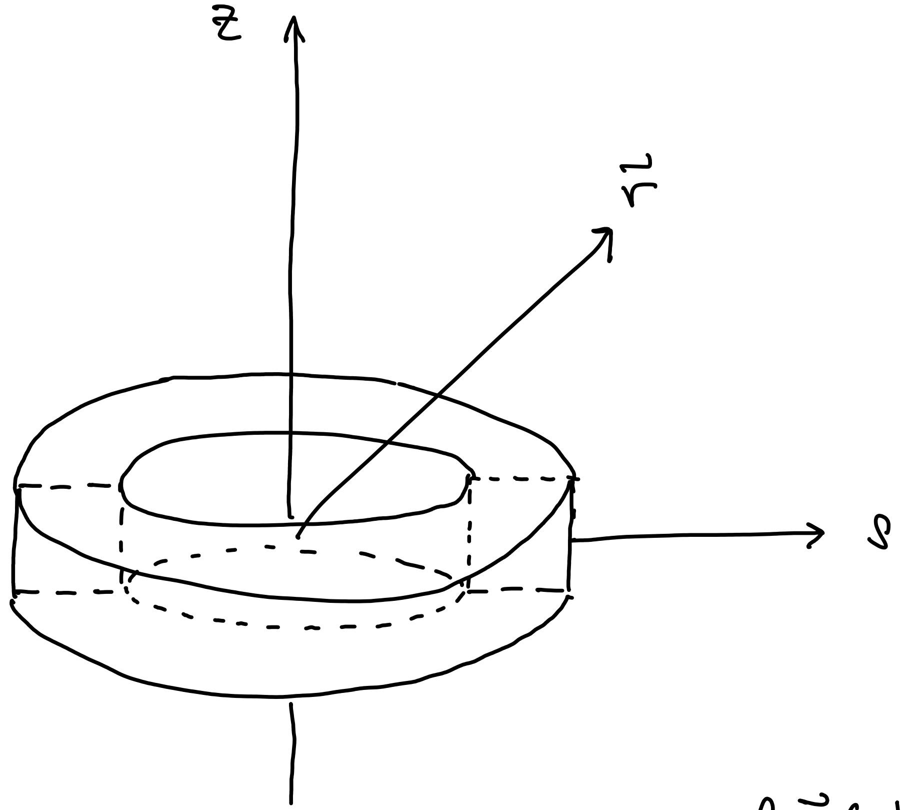
d) electrostatic energy

$$W = \frac{1}{2}\int \vec{D}\cdot\vec{E}\, d^3\vec{r} =$$

$$= 2\pi\left(\frac{Q_f}{4\pi}\right)^2\left[\int_0^a \frac{r}{a^3}\cdot\frac{r}{\varepsilon_0\varepsilon_r a^3} r^2 dr + \int_a^b \frac{1}{r^2}\cdot\frac{1}{\varepsilon_0 r^2} r^2 dr\right] =$$

$$= \frac{Q_f^2}{8\pi\varepsilon_0}\left[\frac{1}{5}\frac{1}{\varepsilon_r a} + \frac{1}{a} - \frac{1}{b}\right]$$

2a)



Biot - Savart: $\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} d^3\vec{r}'$

$$\vec{J}(\vec{r}') = I\left(\hat{z}\,\delta(s'-a) + \hat{s}\,\delta(z'-\tfrac{h}{2}) - \hat{z}\,\delta(s'-b) - \hat{s}\,\delta(z'+\tfrac{h}{2})\right)$$
$$= I\left[\hat{z}\,(\delta(s'-a) - \delta(s'-b)) + (\hat{x}\cos\varphi' + \hat{y}\sin\varphi')\left(\delta(z'-\tfrac{h}{2}) - \delta(z'+\tfrac{h}{2})\right)\right]$$

choose coordinate system so that $\vec{r} = z\hat{z} + s\hat{x}$

$$\vec{r} - \vec{r}' = z\hat{z} + s\hat{x} - z'\hat{z} - s'\cos\varphi'\hat{x} - s'\sin\varphi'\hat{y}$$

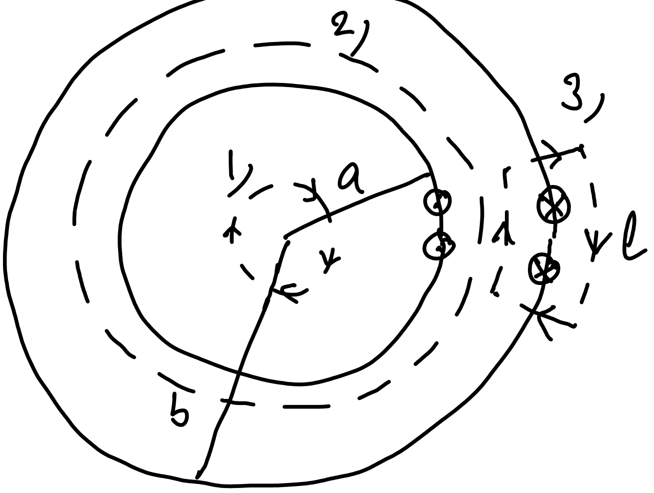
the cross-product becomes

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \cos\varphi'\left(\delta(z'-\frac{h}{2}) - \delta(z'+\frac{h}{2})\right) & \sin\varphi'\left(\delta(z'-\frac{h}{2}) + \delta(z'+\frac{h}{2})\right) & \delta(s'-a) - \delta(s'-b) \\ s - s'\cos\varphi' & s'\sin\varphi' & z - z' \end{vmatrix}$$

after integration over φ' all terms proportional to $\sin\varphi'$ and $\cos\varphi'$ will vanish leaving

$$\int \left(\vec{J}(\vec{r}') \times (\vec{r} - \vec{r}')\right) d\varphi' = I\left(\delta(s'-a) - \delta(s'-b)\right) s\,\hat{y} \sim \hat{\varphi}$$

b)



Ampere's law: $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \Rightarrow \oint \vec{B} \cdot d\vec{l} = \mu_0 \int \vec{J} \cdot d\vec{S}$

Applying Ampere's law on the path 1) for any z

$$\int \vec{B} \cdot d\vec{l} = 0 \quad [\vec{B} \propto \hat{\varphi}] \Rightarrow \vec{B}(s<a, \varphi, z) = 0$$

the same arg. gives $\vec{B}(s>b, \varphi, z) = 0$

The rectangular shape of the coil gives for $a<s<b$ and $-\frac{h}{2} < z < \frac{h}{2}$ using path 2)

$$\int \vec{B} \cdot d\vec{l} = NI \quad [\vec{B} \sim \hat{\varphi}] \Rightarrow \vec{B} = \mu_0 \frac{NI}{2\pi s} \hat{\varphi}$$

outside the coil $\vec{B} = 0$ again.

For any other shape of the coil one has to choose the amperian loop similar to path 3).

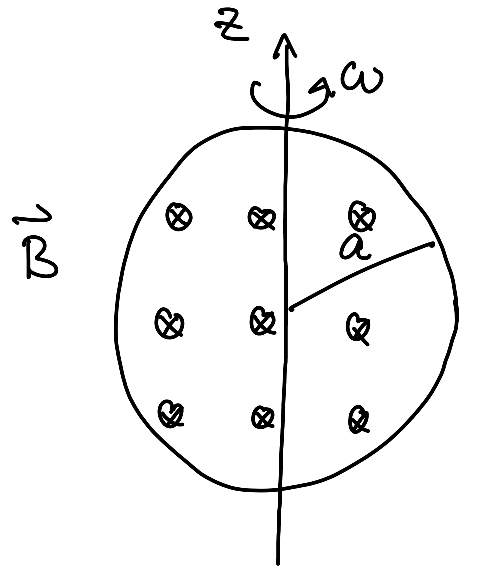
c) The flux $\Phi = \int \vec{B} \cdot d\vec{S} \Rightarrow$

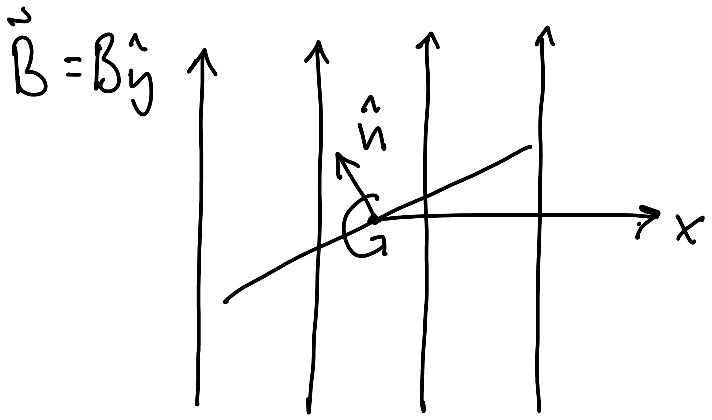
$$\Phi_{1\text{-turn}} = \mu_0 \frac{NI}{2\pi} h \int_a^b \frac{1}{s} ds = \frac{\mu_0 NI h}{2\pi} \ln \frac{b}{a}$$

$$\Phi_{\text{toroid}} = N \Phi_{1\text{-turn}}$$

Inductance $\Phi = LI \Rightarrow L = \frac{\mu_0 N^2 h}{2\pi} \ln \frac{b}{a}$

3.





choose z-axis along rotation axis, y-axis along $\vec{B}$

normal vector of surface spanned by ring:

$$\hat{n} = \hat{y}\cos\omega t - \hat{x}\sin\omega t$$

a) the flux $\Phi = \int \vec{B}\cdot d\vec{S} = \pi a^2 B \cos\omega t$

b) Faradays law $\vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}\vec{B} \Rightarrow \oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\Phi$

$\vec{E}$-field along ring

$$2\pi a E = \pi a^2 B \sin\omega t \Rightarrow E = \frac{aB\omega}{2}\sin\omega t$$

c) power density $\frac{\partial}{\partial t} w = \vec{J}\cdot\vec{E}$

Ohm's law $\vec{J} = \sigma\vec{E}$

$$\left.\begin{matrix} \frac{\partial}{\partial t} w = \vec{J}\cdot\vec{E} \\ \vec{J} = \sigma\vec{E} \end{matrix}\right\} \frac{\partial}{\partial t} w = \frac{\sigma a^2 B^2 \omega^2}{4}\sin^2\omega t$$

Integrating over the ring

$$P = \int \vec{J}\cdot\vec{E}\, d^3\vec{r} = \frac{\sigma a^2 B^2 \omega^2}{4}\sin^2\omega t \; 2\pi a A$$

$$\left[R = \frac{2\pi a}{\sigma A}\right] = \frac{\pi^2 a^4 B^2 \omega^2 \sin^2\omega t}{R} = RI^2$$

d) magnetic dipolemoment

$$\vec{m} = I\pi a^2 \hat{n}, \quad I = JA = \frac{\pi a^2 B\omega \sin\omega t}{R}$$

4. Three-vector potential

$$\vec{\tilde{A}}_{md}(\vec{r},t) = \frac{ik\mu_0}{4\pi}(\hat{r}\times\vec{\tilde{m}}_0)\frac{e^{i(kr-\omega t)}}{r}$$

choose $\vec{\tilde{m}}_0 = \tilde{m}_0\hat{z} = \tilde{m}_0(\hat{r}\cos\theta - \hat{\theta}\sin\theta)$

$$\Rightarrow \vec{\tilde{A}}_{md} = -\frac{ik\mu_0\tilde{m}_0}{4\pi}\sin\theta\,\frac{e^{i(kr-\omega t)}}{r}\hat{\varphi} = \tilde{A}_{md}\hat{\varphi}$$

a) homogeneous wave-eqn (for each component)

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right)\vec{\tilde{A}}_{md} = 0$$

inserting $\tilde{A}_{md}$ gives

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{\tilde{A}}_{md} = -\frac{\omega^2}{c^2}\vec{\tilde{A}}_{md}$$

$$\vec{\nabla}^2\left(-\sin\theta\frac{e^{ikr}}{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(-\sin\theta\frac{e^{ikr}}{r}\right)\right) +$$

$$+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\left(-\sin\theta\frac{e^{ikr}}{r}\right)\right) + 0 =$$

$$= -\frac{\sin\theta}{r^2}\frac{\partial}{\partial r}\left(r^2\left(\frac{ik}{r}e^{ikr} - \frac{1}{r^2}e^{ikr}\right)\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(-\sin\theta\cos\theta)\frac{e^{ikr}}{r}$$

$$[kr \gg 1] = -\frac{\sin\theta}{r}k^2 e^{ikr}$$

$$\Rightarrow \vec{\nabla}^2\vec{\tilde{A}}_{md} = -k^2\vec{\tilde{A}}_{md} = -\frac{\omega^2}{c^2}\vec{\tilde{A}}_{md} \qquad \square$$

Radiation zone $kr \gg 1 \Leftrightarrow r \gg \lambda$

$\Rightarrow$ very far from source and the fields are to a good approximation planar

b) Lorentz gauge: $\frac{1}{c^2}\frac{\partial}{\partial t}\tilde{V}_{md} + \vec{\nabla}\cdot\vec{\tilde{A}}_{md} = 0$

$$\left.\begin{aligned}&\vec{\nabla}\cdot\vec{\tilde{A}}_{md} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\tilde{A}_{md} = 0 \Rightarrow \frac{\partial}{\partial t}\tilde{V}_{md} = 0\\ &\tilde{V}_{md} = \tilde{V}_{md,0}\,e^{-i\omega t} \Rightarrow \frac{\partial}{\partial t}\tilde{V}_{md} = -i\omega\tilde{V}_{md}\end{aligned}\right\}\ \tilde{V}_{md} = 0$$

c) write $\tilde{\vec{A}}_{md} = \underbrace{-\frac{ik\mu_0 \tilde{m}_0}{4\pi}}_{C} \sin\theta \, \frac{e^{i(kr-\omega t)}}{r} \, \hat{\varphi}$

$$\tilde{\vec{E}} = -\vec{\nabla}\tilde{V}_{md} - \frac{\partial}{\partial t}\tilde{\vec{A}}_{md} = \overbrace{i\omega C}^{\frac{k\omega\mu_0\tilde{m}_0}{4\pi}} \sin\theta \, \frac{e^{i(kr-\omega t)}}{r} \, \hat{\varphi}$$

$$\tilde{\vec{B}} = \vec{\nabla}\times\tilde{\vec{A}}_{md} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \, C \sin\theta \, \frac{e^{i(kr-\omega t)}}{r}\right)\hat{r}$$

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(C \sin\theta \, e^{i(kr-\omega t)}\right)\hat{\theta} = \left[\frac{1}{r^2} << \frac{k}{r}\right]$$

$$= \underbrace{-ikC}_{-\frac{k^2\mu_0\tilde{m}_0}{4\pi}} \sin\theta \, \frac{e^{i(kr-\omega t)}}{r}\,\hat{\theta} = \left[k = \frac{\omega}{c}\right] = \frac{1}{c}\,\hat{r}\times\tilde{\vec{E}}$$

d) time average of radiated intensity given by

$$\langle I\rangle = \frac{1}{2}\frac{1}{\mu_0}\left|\tilde{\vec{E}}\times\tilde{\vec{B}}^*\right| = \omega k |C|^2 \sin^2\theta \, \frac{1}{r^2}\,\hat{r}$$

Total radiated power

$$\langle P\rangle = \frac{1}{2\mu_0}\int I \, r^2 d(\cos\theta)\, d\varphi = \frac{\omega k |C|^2}{2\mu_0}\underbrace{\int_{-1}^{1}(1-\cos^2\theta)\, d(\cos\theta)}_{4/3}$$

$$= \frac{4\pi\omega k |C|^2}{3\mu_0} = \frac{\mu_0 \omega k^3 |\tilde{m}_0|^2}{12\pi}$$