Some answers to exam 2014-05-30

1. Use boundary conditions to determine the coefficients in the general solution

$$V(r,\theta) = \sum_l \left(a_l r^l + \frac{b_l}{r^{l+1}} \right) P_l(\cos\theta)$$

Boundary cond: $V_{in} = V_{out}$

$$\varepsilon_{in} \frac{\partial V}{\partial r}\Big|_{in} - \varepsilon_{out} \frac{\partial V}{\partial r}\Big|_{out} = -\sigma_f = 0$$

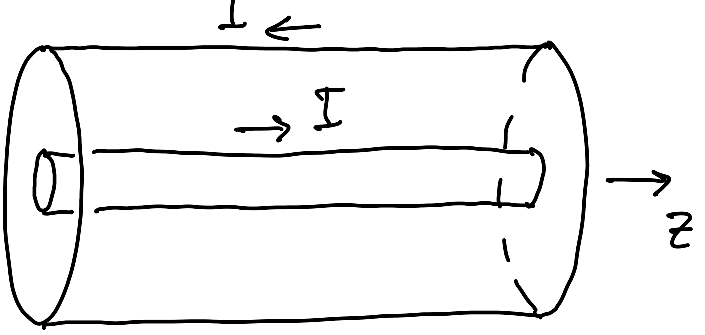
$$V_{out} \to -E_0 r\cos\theta \quad \text{as } r \to \infty$$

b) $\vec{P} = \chi_e \vec{E} = (\varepsilon_r - 1)\vec{E}$

$\vec{p} = \int \vec{P} \, d^3\vec{r}$

c) $\sigma_b = \vec{P} \cdot \hat{n} = \vec{P} \cdot \hat{r}$

2 a) Ampere's law + right hand rule



Between currents

$$\int \vec{H} \cdot d\vec{l} = I$$

$$\Rightarrow \vec{H} = \frac{I}{2\pi s} \hat{\varphi}$$

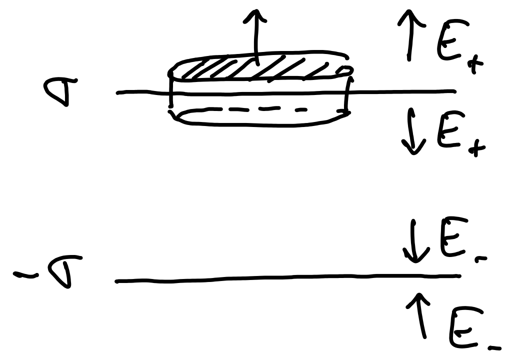
$$\vec{B} = \mu \vec{H} = \mu_0 (1 + \chi_m) H$$

b) $\vec{K}_b = \vec{M} \times \hat{n}$, $\vec{K}_b(s=a) = \chi_m \frac{I}{2\pi a} \hat{z}$

c) $W = \frac{1}{2} \int \vec{H} \cdot \vec{B} \, d^3\vec{r} \Rightarrow \frac{W}{l} = \frac{\mu}{4\pi} I^2 \ln\frac{b}{a}$

$W = \frac{1}{2} L I^2 \Rightarrow \frac{L}{l} = \frac{\mu}{2\pi} \ln\frac{b}{a}$

3 a) Gauss's law $\int \vec{E}\cdot d\vec{S} = \frac{1}{\varepsilon_0}\int \rho\, d^3r = \frac{1}{\varepsilon_0}\int \sigma\, dA$

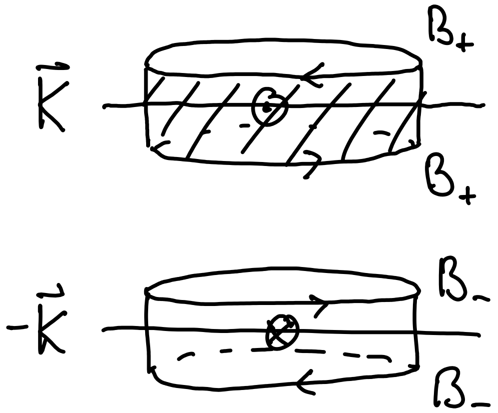


$2E_+ dA = \frac{1}{\varepsilon_0}\sigma\, dA$, $E_+ = \frac{1}{2}\frac{\sigma}{\varepsilon_0}$

$E_- = \frac{1}{2}\frac{\sigma}{\varepsilon_0}$, $\sigma = \frac{It}{\pi a^2}$

$E_{in} = E_+ + E_- = \frac{\sigma}{\varepsilon_0}$, $E_{out} = E_+ - E_- = 0$

b) Ampere's law $\int \vec{B}\cdot d\vec{l} = \mu\int\left(\vec{J} + \varepsilon\frac{\partial}{\partial t}\vec{E}\right)\cdot d\vec{S}$



$2B_+ 2\pi s = \mu\, \vec{K}\cdot\hat{s}\, 2\pi s = \mu I \frac{s^2}{a^2}$

$\Rightarrow B_+ = \mu\frac{Is}{4\pi a^2}$

$B_- = B_+$

$\vec{B}_{in} = (B_+ + B_-)\hat{\varphi} = \mu\frac{Is}{2\pi a^2}\hat{\varphi}$, $\vec{B}_{out} = 0$

c) $\frac{\partial}{\partial t}\vec{D} = \frac{I}{\pi a^2}\hat{z}$

$\hat{z} \perp d\vec{S} \Rightarrow$ no contribution in b)

4 a) ME: $\vec{\nabla}\cdot\vec{E} = 0$ $\quad \vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}\vec{B}$

$\vec{\nabla}\cdot\vec{B} = 0$ $\quad \vec{\nabla}\times\vec{B} = \mu\vec{J} + \mu\varepsilon\frac{\partial}{\partial t}\vec{E}$

Ohm's law: $\vec{J} = \sigma\vec{E}$

$\Rightarrow \vec{\nabla}^2\vec{E} = \mu\varepsilon\frac{\partial^2}{\partial t^2}\vec{E} + \mu\sigma\frac{\partial}{\partial t}\vec{E}$

Inserting solution $\Rightarrow \tilde{k}^2 = \mu\left(\varepsilon + \frac{i\sigma}{\omega}\right)\omega^2$

Faraday's law $\Rightarrow$ $\tilde{\vec{B}}(\vec{r},t) = \frac{1}{\omega}\,\tilde{\vec{k}} \times \tilde{\vec{E}}(\vec{r},t)$

b) $\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{S} \Rightarrow \vec{E}_1^{\parallel} = \vec{E}_2^{\parallel}$

$\oint \vec{H}\cdot d\vec{l} = \int\left(\varepsilon \frac{\partial}{\partial t}\vec{E} + \sigma\vec{E}\right)\cdot d\vec{S} \Rightarrow \vec{H}_1^{\parallel} = \vec{H}_2^{\parallel}$

c) $\vec{E}$ cont: $\tilde{E}_i + \tilde{E}_r = \tilde{E}_t$

$\vec{H}$ cont: $\frac{n_1}{c\mu_0}(\tilde{E}_i - \tilde{E}_r) = \frac{\tilde{n}_2}{c\mu_0}\tilde{E}_t$

5 a) $\vec{A}_{ed} = -i\omega \frac{\mu_0}{4\pi}\frac{1}{r}e^{i(kr-\omega t)}\,\vec{p}_0$

$\vec{A}_{tot} = i\omega\frac{\mu_0}{4\pi}\vec{p}_0\left[\frac{1}{r_+}e^{ikr_+} - \frac{1}{r_-}e^{ikr_-}\right]e^{-i\omega t}$

$r_\pm = r\left(1 \mp \frac{a}{2r}\cos\theta\right)$

$e^{ikr_\pm} = e^{ikr}\left(1 \mp i\frac{ka}{2}\cos\theta\right)$

$\Rightarrow \vec{A}_{tot} = \omega a k \cos\theta\,\frac{\mu_0 p_0}{4\pi r}\,e^{i(kr-\omega t)}\,\hat{z}$

b) $\vec{B} = \vec{\nabla}\times\vec{A} = [kr \gg 1] =$

$= \omega a k\cos\theta\,\frac{\mu_0 p_0}{4\pi r}(ik)e^{i(kr-\omega t)}\underbrace{\hat{r}\times\hat{z}}_{-\sin\theta\,\hat{\varphi}}$

$\vec{E} = \frac{ic^2}{\omega}\vec{\nabla}\times\vec{B} =$

$= \frac{ic^2}{\omega}\omega k a\cos\theta\,\frac{\mu_0 p_0}{4\pi r}(ik)^2 e^{i(kr-\omega t)}\sin\theta\,\hat{\theta}$

c) $\langle\vec{S}\rangle = \frac{1}{2\mu_0}\vec{E}\times\vec{B}^* =$

$= \frac{\mu_0 p_0^2 a^2 \omega^6 \sin^2 2\theta}{128\pi^2 c^3 r^2}\,\hat{r}$