Electrostatic fields in matter:

conductors — insulators / dielectrics

unlimited supply of free charges — all charges attached to specific atom/molecule

$\Rightarrow \vec{E}_{in} = 0$ — $\Rightarrow \vec{E}_{in} = ?$

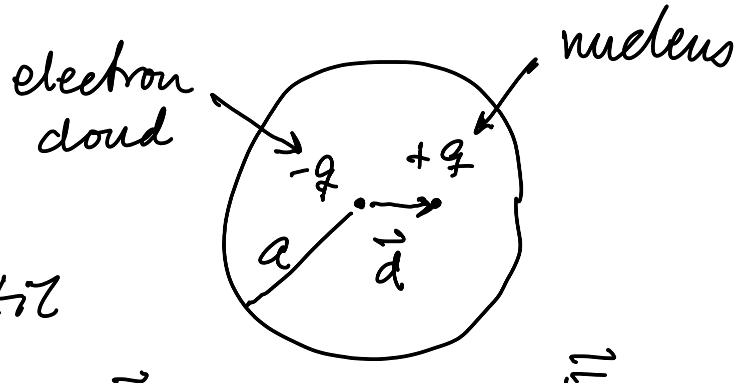
model [uncharged] dielectric material as a collection of dipoles

$$\vec{p} = \int \vec{r}\,' \rho(\vec{r}\,')\, d^3\vec{r}\,' \qquad \left(Q = \int \rho(\vec{r}\,') d^3\vec{r}\,' = 0\right)$$

need to know more about physical dipoles in $\vec{E}$-fields

1) <u>Polarizability</u>

$$\vec{p} = \alpha \vec{E}$$



atom stretched until

$$\vec{E}_e = -\frac{1}{4\pi\varepsilon_0} \frac{q\vec{d}}{a^3} = -\vec{E}$$

field inside uniformly charged sphere

$$\Rightarrow \vec{p} = q\vec{d} = \underbrace{4\pi\varepsilon_0 a^3}_{\alpha} \vec{E}$$

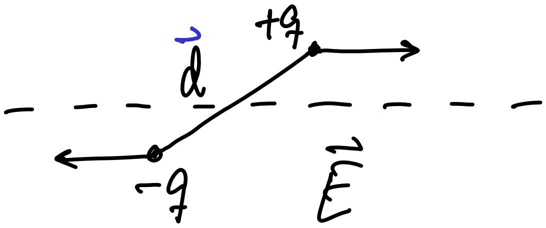
[for molecules $\quad p_i = \sum_j \alpha_{ij} E_j$]

<u>note</u>: $\vec{E}$ <u>not</u> only external field

2) Alignment

What if molecule has a dipole moment (H_2O)

Uniform $\vec{E}$-field gives torque



$$\vec{N} = \vec{p} \times \vec{E}$$

Varying field gives net force ($|\vec{d}| \to 0$)

$$\vec{F} = q(\vec{d} \cdot \vec{\nabla})\vec{E} = (\vec{p} \cdot \vec{\nabla})\vec{E}$$

All in all we see that an $\vec{E}$-field will create / line-up dipoles in the "same" direction – called <u>polarisation</u>

$$\vec{P} = \frac{\text{dipole moment}}{\text{unit volume}}$$

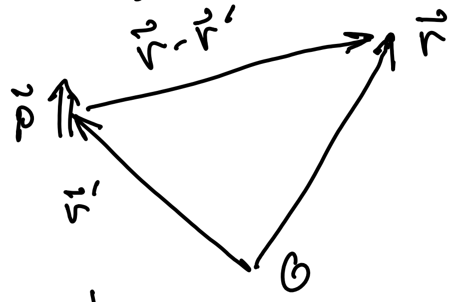
Two steps:

(i) what are the effects of the polarisation?

(ii) what is the cause of the polarisation?

Electric field from polarisation?

Know



$$V_{dip}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{\vec{p} \cdot (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$

$$\Rightarrow V_{pol}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\vec{P}(\vec{r}') \cdot (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \, d^3\vec{r}'$$

physical meaning?

Rewrite using $\vec{\nabla}'\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) = \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$

derivative wrt $\vec{r}'$

$$\Rightarrow V_{pol}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \vec{P}(\vec{r}')\cdot\vec{\nabla}'\frac{1}{|\vec{r}-\vec{r}'|}\, d^3\vec{r}' =$$

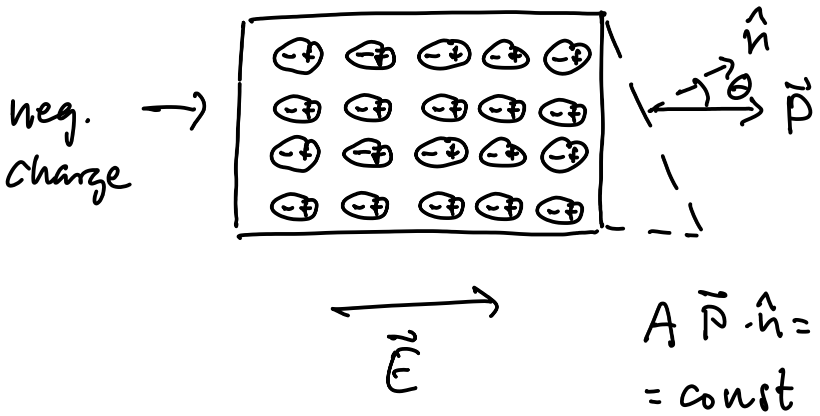
$$= \frac{1}{4\pi\varepsilon_0}\int_V \left[\vec{\nabla}'\cdot\left(\frac{\vec{P}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right) - \frac{1}{|\vec{r}-\vec{r}'|}\vec{\nabla}\cdot\vec{P}(\vec{r}')\right] d^3\vec{r}'$$

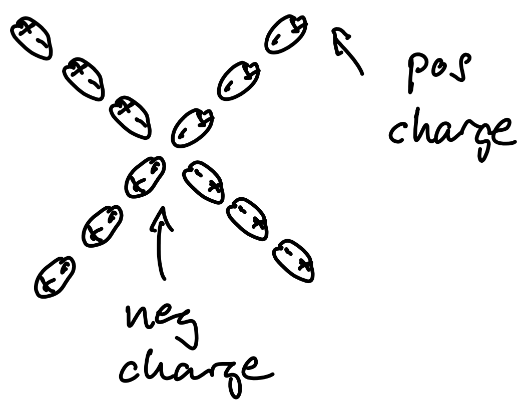
$$= \frac{1}{4\pi\varepsilon_0}\oint_S \frac{\vec{P}(\vec{r}')}{|\vec{r}-\vec{r}'|}\cdot d\vec{S}' - \frac{1}{4\pi\varepsilon_0}\int_V \frac{1}{|\vec{r}-\vec{r}'|}\underbrace{\vec{\nabla}\cdot\vec{P}(\vec{r}')}_{-\rho_b(\vec{r}')}\, d^3\vec{r}'$$

$\vec{P}(\vec{r}')\cdot\hat{n} = \sigma_b$

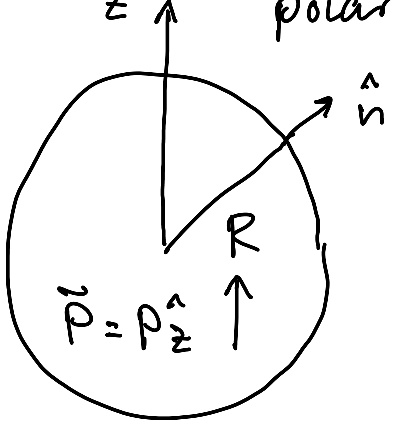
$$= \frac{1}{4\pi\varepsilon_0}\oint_S \frac{\sigma_b}{|\vec{r}-\vec{r}'|}\, dS' + \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho_b(\vec{r}')}{|\vec{r}-\vec{r}'|}\, d^3\vec{r}'$$

$\therefore$ potential of bound surface + volume charge





Example: electric field from uniformly polarised sphere of radius R



$\sigma_b = \vec{P}\cdot\hat{n} = P\cos\theta$

$\rho_b = -\vec{\nabla}\cdot\vec{P} = 0$

brute force: $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{P\hat{z}\cdot(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\, d^3\vec{r}\,'$

quite painful

instead use separation of variables. In fact same charge distr. as conducting sphere in external field only boundary cond. different: $V \to 0$ as $r \to \infty$ instead of $V \to -E_0 r\cos\theta$ so add $E_0 r\cos\theta$ to V_{cs} with $E_0 = \frac{P}{3\varepsilon_0}$

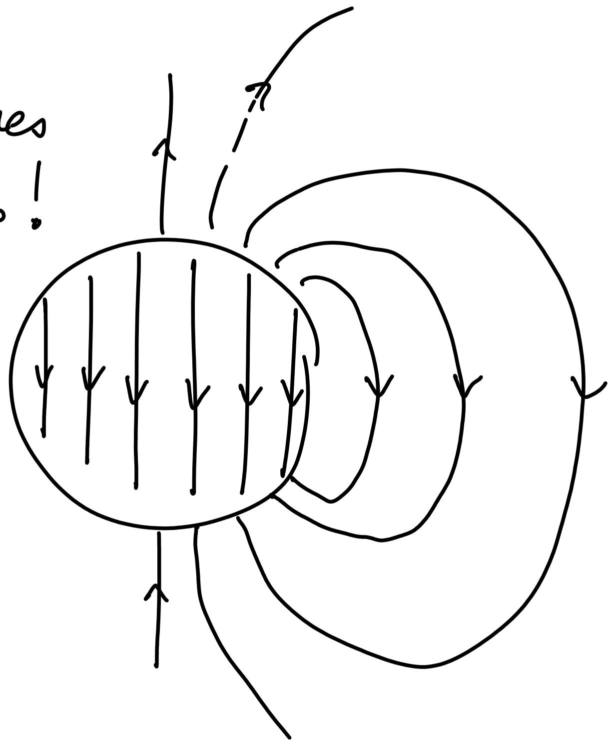
$$\Rightarrow V(r,\theta) = \begin{cases} \frac{Pr}{3\varepsilon_0}\cos\theta, & r \le R \\ \underbrace{\frac{PR^3}{3\varepsilon_0}\frac{1}{r^2}\cos\theta}, & r \ge R \end{cases}$$

$$\frac{1}{4\pi\varepsilon_0}\underbrace{\frac{4\pi R^3}{3}\vec{P}}_{\vec{p}}\cdot\frac{\hat{r}}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} = V_{dip.}$$

and the electric field inside the sphere is given by $(r\cos\theta = z)$

$$\vec{E} = -\vec{\nabla}V = -\frac{1}{3\varepsilon_0}\vec{P}, \qquad r < R$$

note: field lines start/end on σ_b!



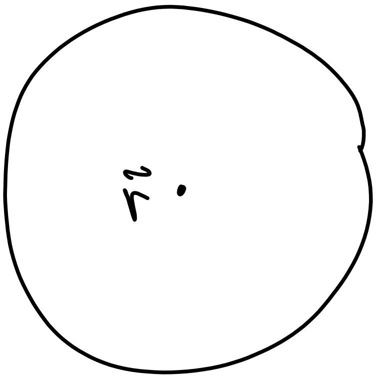
The electric field inside a dielectric - take 2:

Possible problems:
- used potential of perfect dipole
- assumed $\vec{P}$ is continuous

not valid on atomic level

Solution - need to average over large enough chunks of matter (thousands of atoms) to make field continuous but still small compared to object of interest

Consider "small" sphere around a point of interest inside matter



The total field in point

$$\langle \vec{E} \rangle = \langle \vec{E}_{in} \rangle + \langle \vec{E}_{out} \rangle$$

$\uparrow$ from within sphere $\quad$ $\uparrow$ from outside sphere

Problem 3.47: the average field from all charges outside sphere is the same as the field they produce at the center

Make sphere "large" enough so that we can use continuous $\langle \vec{P}(\vec{r}\,')\rangle$ outside

$$\Rightarrow V_{out}(\vec{r}) = \langle V_{out}(\vec{r})\rangle = \frac{1}{4\pi\varepsilon_0}\int_{\text{outside}} \frac{\langle \vec{P}(\vec{r}\,')\rangle \cdot (\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\, d^3\vec{r}\,'$$

Also need the average field from all charges inside the sphere, $\langle \vec{E}_{in}(\vec{r})\rangle$

Problem 3.47:

$$\langle \vec{E}_{in}(\vec{r})\rangle = -\frac{1}{4\pi\varepsilon_0}\frac{\vec{p}}{R^3}$$

where $\vec{p}$ is the total dipolmoment

by definition $\vec{p} = \frac{4\pi R^3}{3}\langle \vec{P}(\vec{r})\rangle$

$$\Rightarrow \langle \vec{E}_{in}(\vec{r})\rangle = -\frac{1}{3\varepsilon_0}\langle \vec{P}(\vec{r})\rangle$$

but this is just the field from a uniformly polarized sphere

$$\langle V_{in}(\vec{r})\rangle = \frac{1}{4\pi\varepsilon_0}\int_{\text{inside}} \frac{\langle \vec{P}(\vec{r}\,')\rangle \cdot (\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\, d^3\vec{r}\,'$$

$$\therefore \langle V(\vec{r})\rangle = \frac{1}{4\pi\varepsilon_0}\int \frac{\langle \vec{P}(\vec{r}\,')\rangle \cdot (\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\, d^3\vec{r}\,'$$

from now on – drop $\langle\ \rangle$ as in book.

The displacement field

The polarisation gives rise to an $\vec{E}$-field which in turn affects the polarisation - how to break this catch-22?

First step: introduce $\vec{D}$-field

Polarisation produces bound surface and volume charges - but there may also be additional free charges within or outside the dielectric

The source of the total electric field is the total charge. In the dielectric we have

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\varepsilon_0} \rho = \frac{1}{\varepsilon_0} (\rho_f + \rho_b) = \frac{1}{\varepsilon_0} (\rho_f - \vec{\nabla} \cdot \vec{P})$$

$$\Rightarrow \quad \vec{\nabla} \cdot \underbrace{(\varepsilon_0 \vec{E} + \vec{P})}_{\vec{D}} = \rho_f$$

Gauss' law can be written

$$\vec{\nabla} \cdot \vec{D} = \rho_f \quad , \quad \oint \vec{D} \cdot d\vec{S} = Q_{f,enc}$$

Warning:

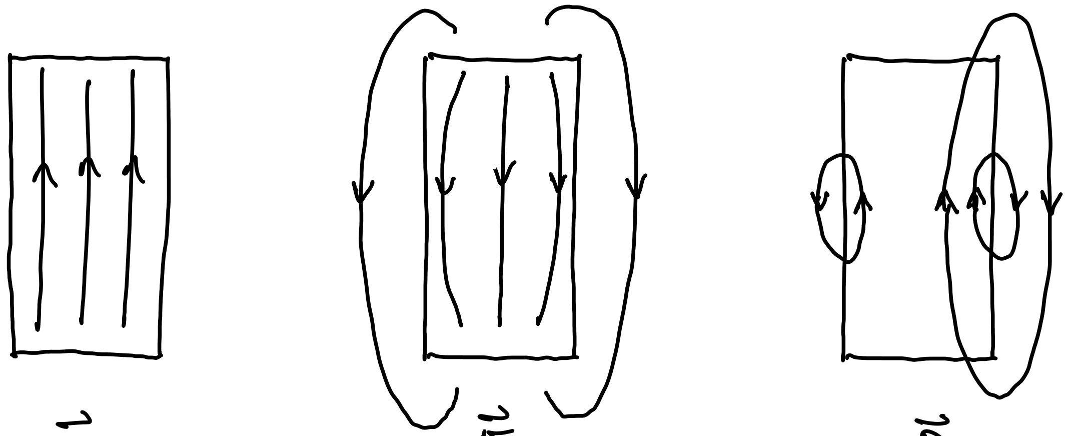
$$\vec{\nabla} \times \vec{D} = \varepsilon_0 \underbrace{\vec{\nabla} \times \vec{E}}_{=0} + \underbrace{\vec{\nabla} \times \vec{P}}_{\text{may be nonzero!}} \neq 0$$

Example: Consider a cylinder with radius a and length L = 2a carrying a "frozen-in" polarization $\vec{P}$ parallel to its axis. Find the bound charge and sketch $\vec{P}$, $\vec{E}$ and $\vec{D}$.

$$\vec{P} = P\hat{z} = \text{const}$$

$$\Rightarrow \quad \vec{\nabla}\cdot\vec{P} = 0 \quad \Rightarrow \quad \rho_b = 0$$

$$\vec{P}\cdot\hat{n} = \pm P \quad \Rightarrow \quad \sigma_b = \pm P$$



$\vec{P}$

$\vec{E}$

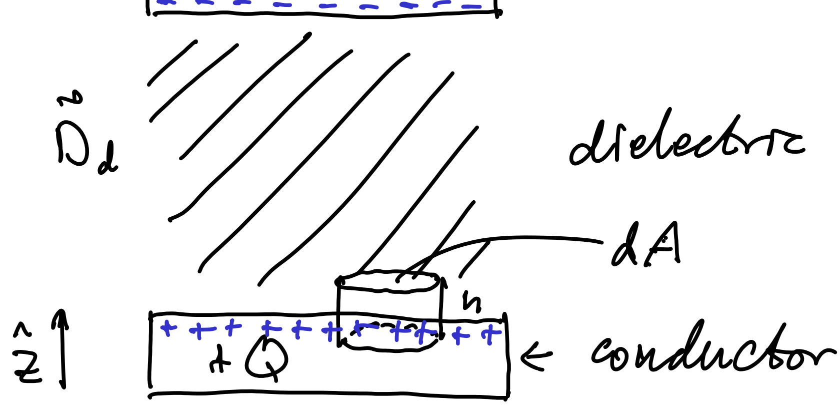
($\vec{E}$ starts/ends on charges)

$\vec{D}$

($\vec{D}$ starts/ends on free charges)

$\rho_f = 0$

Example: plate capacitor



[all charges on surfaces of conductors facing each other]

Gauss' law on boundary between conductor and dielectric

$$\oint \vec{D}\cdot d\vec{S} = \int_V \rho_f \, d^3\vec{r}$$

$$\vec{D}_d \cdot \hat{z}\, dA - \vec{D}_c \cdot \hat{z}\, dA = \sigma_f \, dA \;, \quad \sigma_f = \frac{Q}{A}$$

no $\vec{D}$-field in conductor

$$\Rightarrow \vec{D}_d = \sigma_f \hat{z} \quad \text{in dielectric}$$

but to find out $\vec{E}$ we need $\vec{P}$ and for this we have to make some assumptions

constant, linear dependence!, set to zero

$$\vec{P} = \varepsilon_0 \chi_e \vec{E} + \text{non-}\cancel{\text{linear}}$$

electric susceptibility; total $\vec{E}$-field (including $\vec{P}$!)

$$\Rightarrow \vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \underbrace{(1+\chi_e)}_{\varepsilon_r} \vec{E} = \varepsilon \vec{E}$$

relative permittivity or dielectric constant; permittivity

χ_e same in all directions

So for linear, isotropic and homogeneous materials we have

χ_e same in all points

$$\vec{D} = \varepsilon \vec{E}$$

Boundary value problems

In special case of linear, isotropic, and homogeneous dielectrics we have $\rho_b \propto \rho_f$:

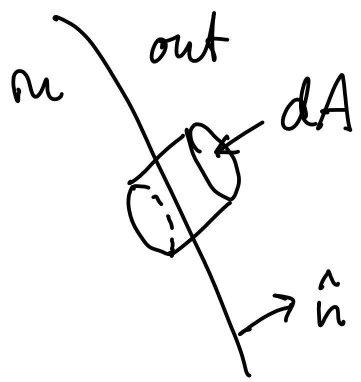
$$\vec{P} = \vec{D} - \varepsilon_0 \vec{E} = \vec{D} - \frac{\varepsilon_0}{\varepsilon}\vec{D} = \frac{\varepsilon_0}{\varepsilon}\chi_e \vec{D} = \frac{\chi_e}{1+\chi_e}\vec{D}$$

$$\Rightarrow \quad \rho_b = -\vec{\nabla}\cdot\vec{P} = -\frac{\chi_e}{1+\chi_e}\underbrace{\vec{\nabla}\cdot\vec{D}}_{\rho_f}$$

So for uncharged material, $\rho = \rho_b + \rho_f = 0$ inside*

$\Rightarrow$ can calculate electric potential using separation of variables ($\nabla^2 V = 0$)

boundary conditions:



$$\oint \vec{D}\cdot d\vec{S} = Q_{enc}$$

$$\Rightarrow \quad D^{\perp}_{out} - D^{\perp}_{in} = \sigma_f$$

$$\Rightarrow \quad \varepsilon_{out} E^{\perp}_{out} - \varepsilon_{in} E^{\perp}_{in} = \sigma_f$$

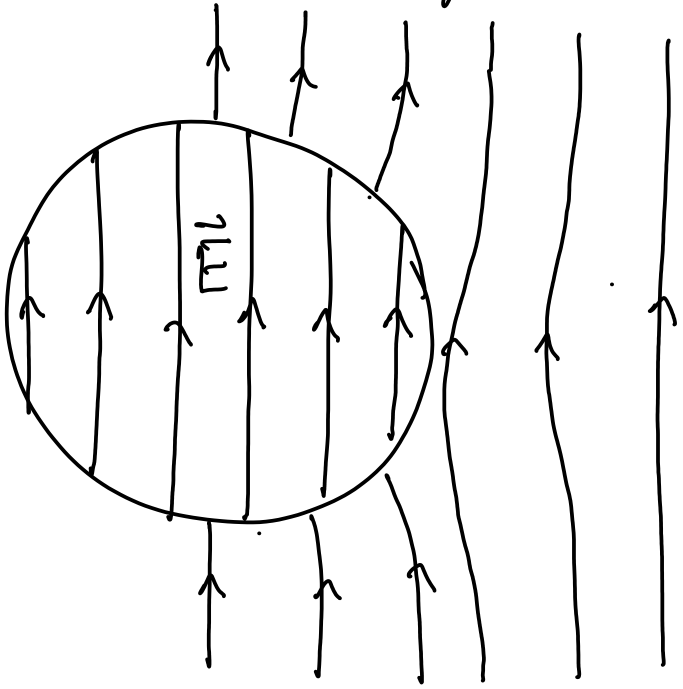
$$\Rightarrow \quad \varepsilon_{out}\frac{\partial V_{out}}{\partial n} - \varepsilon_{in}\frac{\partial V_{in}}{\partial n} = -\sigma_f$$

$$(V_{out} = V_{in})$$

note: $P^{\perp}_{out} - P^{\perp}_{in} = D^{\perp}_{out} - \varepsilon_0 E^{\perp}_{out} - D^{\perp}_{in} + \varepsilon_0 E^{\perp}_{in} = \sigma_f - \sigma = -\sigma_b$

* free charge on surface $\Rightarrow$ bound charge on surface

Example: sphere of linear, isotropic, and homogeneous dielectric material placed in an otherwise uniform electric field



cf. conducting sphere in $\vec{E}_0$

$\vec{E}_0$

Boundary conditions:

(i) $V_{in} = V_{out}$ $\qquad r = R$

(ii) $\varepsilon_{in} \frac{\partial V_{in}}{\partial r} = \varepsilon_0 \frac{\partial V_{out}}{\partial r}$ $\qquad r = R$

(iii) $V_{out} \to -E_0 r \cos\theta$ $\qquad r \to \infty$

$$\Rightarrow \begin{cases} V_{in} = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) \\ V_{out} = \underbrace{-E_0 r \cos\theta}_{\uparrow (iii)} + \sum_{\ell=0}^{\infty} B_\ell \frac{1}{r^{\ell+1}} P_\ell(\cos\theta) \end{cases}$$

(i) $\Rightarrow \sum A_\ell R^\ell P_\ell(\cos\theta) = -E_0 R\cos\theta + \sum \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta)$

$$A_\ell R^{2\ell+1} = B_\ell \qquad \ell \neq 1$$

$$A_1 R^3 = -E_0 R^3 + B_1$$

(ii) $\Rightarrow$ $\displaystyle \varepsilon_{in} \sum_{\ell=0}^{\infty} \ell A_\ell R^{\ell-1} P_\ell(\cos\theta) =$

$$= \varepsilon_0 \left(-E_0 \cos\theta - \sum_{\ell=0}^{\infty} \frac{(\ell+1) B_\ell}{R^{\ell+2}} P_\ell(\cos\theta) \right)$$

$$\Rightarrow \begin{cases} \varepsilon_m \ell A_\ell R^{2\ell+1} = -\varepsilon_0 (\ell+1) B_\ell \quad , \ell \neq 1 \\ \varepsilon_m A_1 R^3 = -\varepsilon_0 E_0 R^3 - \varepsilon_0 2 B_1 \end{cases}$$

Solution:

$$A_1 = -\frac{3\varepsilon_0}{\varepsilon_m + 2\varepsilon_0} E_0 \quad , \quad B_1 = \frac{\varepsilon_{in} - \varepsilon_0}{\varepsilon_m + 2\varepsilon_0} R^3 E_0$$

all other $A_\ell, B_\ell = 0$

use that $\varepsilon_m = \varepsilon_r \varepsilon_0$

$$\Rightarrow \begin{cases} V_m(r,\theta) = -\dfrac{3E_0}{\varepsilon_r + 2} r\cos\theta \\ V_{out}(r,\theta) = -E_0 r \cos\theta \left[1 - \dfrac{\varepsilon_r - 1}{\varepsilon_r + 2} \dfrac{R^3}{r^3} \right] \end{cases}$$

and finally

$$\vec{E}_m = \frac{3}{\varepsilon_r + 2} \vec{E}_0 \longrightarrow \begin{cases} \vec{E}_0 & \text{when } \varepsilon_r \to 1 \\ 0 & \text{when } \varepsilon_r \to \infty \end{cases}$$

Energy stored in dielectric system

Adding free charge $d\rho_f$ to dielectric

$$dW = \int d\rho_f\, V\, d^3\vec{r} = \left[d\rho_f = \vec{\nabla}\cdot d\vec{D}\right]$$

[integrate by parts] $= \int \left[\underbrace{\vec{\nabla}\cdot(d\vec{D}\,V)} - d\vec{D}\cdot(\vec{\nabla}V)\right] d^3\vec{r} =$

$\Rightarrow$ surface integral which vanishes as $r \to \infty$

$$= \int_{\text{all of space}} d\vec{D}\cdot\vec{E}\, d^3\vec{r} \qquad \left[\tfrac{1}{2} d\vec{E}^2 = \vec{E}\cdot d\vec{E}\right]$$

use $\vec{D} = \varepsilon \vec{E} \Rightarrow \frac{1}{2} d(\vec{D}\cdot\vec{E}) = (d\vec{D})\cdot\vec{E}$

$$\therefore \quad W_{l.i.h.} = \frac{1}{2}\int \vec{D}\cdot\vec{E}\, d^3\vec{r} \qquad \text{(free charges)}$$

cf. general formula

$$W = \frac{\varepsilon_0}{2}\int \vec{E}^2 d^3\vec{r} \qquad \text{(all charges)}$$

Ex. Plate capacitor with charge $Q = \sigma_f A$ filled with dielectric material

know $\vec{E} = \frac{1}{\varepsilon_0}\sigma\hat{z} = \left[\sigma = \frac{\sigma_f}{\varepsilon_r}\right] = \frac{\sigma_f}{\varepsilon}\hat{z}$

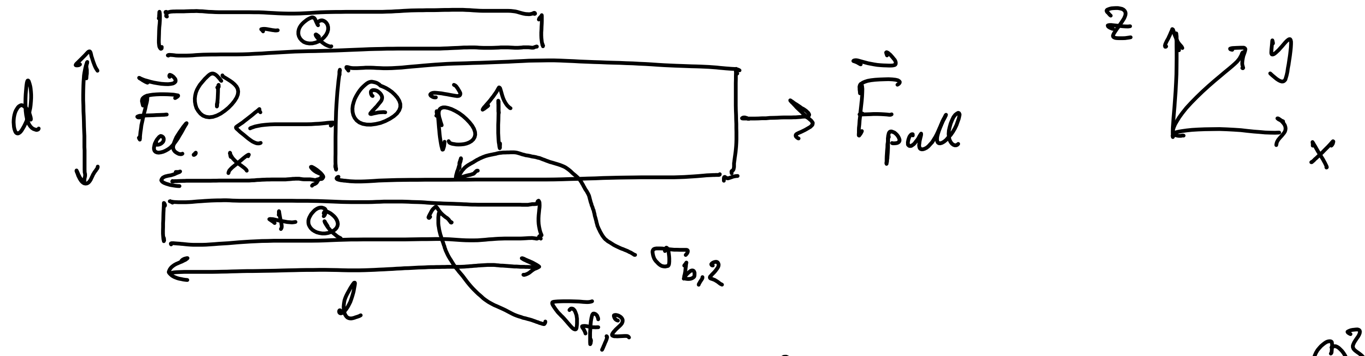
$\vec{D} = \sigma_f \hat{z}$

$$\Rightarrow W = \text{Volume}\cdot\frac{1}{2}\frac{\sigma_f^2}{\varepsilon}$$

cf. without dielectric $(\varepsilon_r = 1)$ $\quad W = \text{Vol}\cdot\frac{1}{2}\frac{\sigma_f^2}{\varepsilon_0}$

Force on dielectric

Consider plate capacitor partially filled with dielectric and Q constant



Force related to energy of capacitor, $W = \dfrac{Q^2}{2C}$

$$dW = \vec{F}_{pull} \cdot d\vec{x} \Rightarrow \vec{F}_{pull} = \frac{dW}{dx}\hat{x} = -\vec{F}_{el}$$

need $W(x)$ or $C(x) = \dfrac{Q}{V(x)}$

conductor $\Rightarrow$ equipot. $V_1 = V_2 = V$

$$\vec{E}_1 = \frac{1}{\varepsilon_0}\sigma_{f,1}\hat{z}, \quad \vec{D}_1 = \sigma_{f,1}\hat{z}, \quad V = \frac{1}{\varepsilon_0}\sigma_{f,1} d$$

$$\vec{D}_2 = \sigma_{f,2}\hat{z}, \quad \vec{E}_2 = \frac{1}{\varepsilon}\sigma_{f,2}\hat{z}, \quad V = \frac{1}{\varepsilon}\sigma_{f,2} d$$

$$\Rightarrow \quad \sigma_{f,1} = \frac{\varepsilon_0}{\varepsilon}\sigma_{f,2} = \frac{1}{\varepsilon_r}\sigma_{f,2}$$

total charge (w = width of cap.)

$$\sigma_{f,1} x w + \sigma_{f,2}(l-x) w = Q$$

$$\sigma_{f,2}\left[\frac{\varepsilon_0}{\varepsilon} x w + (l-x) w\right] = Q$$

$$\therefore \quad \sigma_{f,2} = \frac{\varepsilon Q}{\varepsilon_0 x w + \varepsilon (l-x) w} = \frac{\varepsilon Q}{w(\varepsilon l - (\varepsilon - \varepsilon_0) x)}$$

$$[\varepsilon = \varepsilon_r \varepsilon_0, \ \varepsilon_r = 1 + \chi_e] = \frac{\varepsilon_r Q}{w(\varepsilon_r l - \chi_e x)} = \varepsilon_r \sigma_{f,1}$$

Now the potential

$$V = |\vec{E}_2| d = |\vec{E}_1| d = \frac{\sigma_{f,2}\, d}{\varepsilon_0 \varepsilon_r} = \frac{Q d}{\varepsilon_0 w (\varepsilon_r l - \chi_e x)} = \frac{Q}{C(x)}$$

$$\Rightarrow \quad C(x) = \frac{\varepsilon_0 w}{d} (\varepsilon_r l - \chi_e x)$$

$$\vec{F}_{el} = -\frac{dW}{dx}\hat{x} = +\frac{1}{2}\frac{Q^2}{C^2}\frac{dC}{dx}\hat{x} = -\frac{1}{2}\underbrace{\frac{Q^2}{C^2}}_{V^2}\frac{w\varepsilon_0 \chi_e}{d}\hat{x}$$

$$\left[W = \frac{Q^2}{2C(x)}\right]$$

Physically fringe fields pulling on bound charge in dielectric

$$\vec{P}_2 = \varepsilon_0 \chi_e \vec{E}_2 \Rightarrow \rho_b = -\vec{\nabla}\cdot\vec{P}_2 = 0 \,, \quad \sigma_{b,2} = \vec{P}_2 \cdot \hat{n} = \pm \chi_e \sigma_2$$

check on lower surface of dielectric

$$\sigma_2 = \sigma_{f,2} + \sigma_{b,2} = \sigma_{f,2} - \chi_e \sigma_2 \Rightarrow \sigma_2 (1+\chi_e) = \sigma_{f,2}$$

alt.

$$W = \frac{1}{2}\int \vec{D}\cdot\vec{E}\, d^3\vec{r} = \frac{1}{2}\left[\int_{V_1} \vec{D}_1\cdot\vec{E}_1\, d^3\vec{r} + \int_{V_2} \vec{D}_2\cdot\vec{E}_2\, d^3\vec{r}\right]$$

the total energy is then given by

$$W = \frac{1}{2} d x w \frac{\sigma_{f,1}^2}{\varepsilon_0} + \frac{1}{2} d(l-x) w \frac{\sigma_{f,2}^2}{\varepsilon}$$

$$= \frac{dw}{2}\frac{Q^2}{w^2(\varepsilon_r l - \chi_e x)^2}\underbrace{\left[\frac{x}{\varepsilon_0} + \frac{(l-x)\varepsilon_r^2}{\varepsilon}\right]}_{\frac{1}{\varepsilon_0}\underbrace{(x + (l-x)\varepsilon_r)}_{\varepsilon_r l - \chi_e x}} =$$

$$= \frac{d Q^2}{2\varepsilon_0 w}\frac{1}{\varepsilon_r l - \chi_e x}$$