Electrodynamics

Electromotive force, emf – source of currents in materials

Ohm's law: ↙ force per unit charge

$$\vec{J} = \sigma \vec{f} = \sigma(\vec{E} + \vec{v}\times\vec{B}) \approx \sigma \vec{E}$$

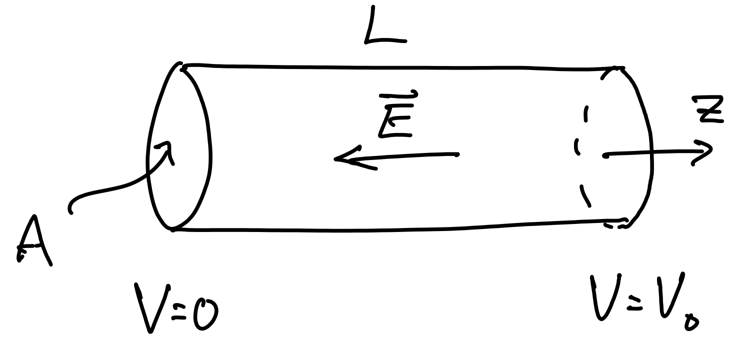
σ ↑ conductivity ($\to \infty$ for perfect. cond.)

for steady currents ($\frac{\partial}{\partial t}\vec{J} = 0$, $\frac{\partial}{\partial t}\rho = 0$) and uniform σ

$$\underset{\text{Gauss's}}{\rho = \varepsilon_0 \vec{\nabla}\cdot\vec{E}} \underset{\text{Ohm's law}}{=} \frac{\varepsilon_0}{\sigma}(\vec{\nabla}\cdot\vec{J}) \underset{\text{cont. eqn}}{=} -\frac{\varepsilon_0}{\sigma}\frac{\partial}{\partial t}\rho = 0$$

∴ all charges on surface, $\vec{\nabla}^2 V = 0$ inside

Ex. 7.3 Cylindrical resistor with cross-sectional area A, length L and conductivity σ. Stipulate potential V constant over each end. What is I?



$$\vec{\nabla}^2 V = 0 + \text{B.C.}$$

$$\Rightarrow V = V_0 \frac{z}{L}$$

$$\vec{E} = -\frac{V_0}{L}\hat{z}$$

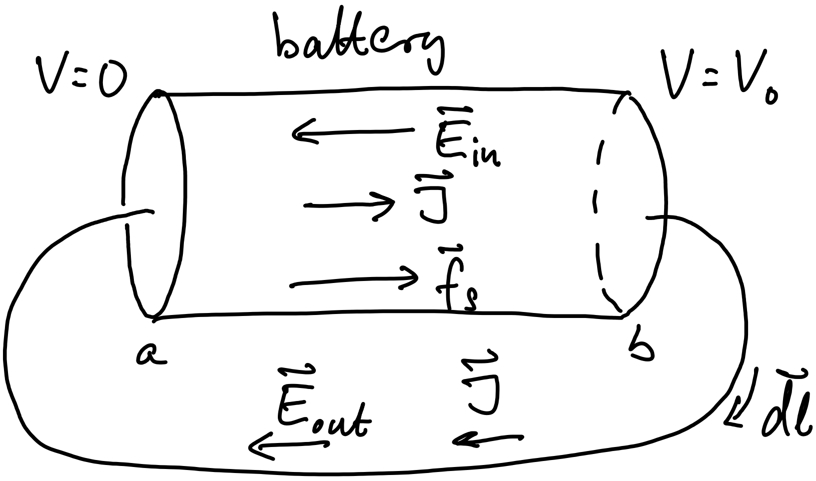
the current $I = JA = \sigma E A = \frac{\sigma A}{L} V_0 = \frac{V_0}{R}$

resistance $R = \frac{L}{\sigma A}$

Current driven by "electromotive force", emf

$$\mathcal{E} \equiv \oint \vec{f}\cdot d\vec{l}$$

simplest example – resistanceless battery:



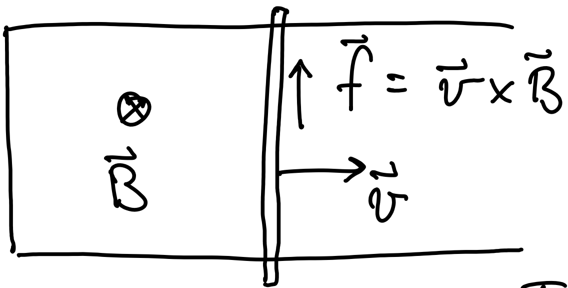
inside: $R=0 \Rightarrow \sigma_{in}=\infty$

$$\Rightarrow \vec{f}=\vec{f}_s+\vec{E}_{in}=\frac{\vec{J}}{\sigma_{in}}=0$$

outside: $\vec{f}=\vec{E}_{out}=\frac{\vec{J}}{\sigma_{out}}$

$$\mathcal{E}\equiv\oint\vec{f}\cdot\vec{dl}=\Big[\underset{\text{static!}}{\oint\vec{E}\cdot\vec{dl}=0}\Big]=\int_a^b\vec{f}_s\cdot\vec{dl}=-\int_a^b\vec{E}_{in}\cdot\vec{dl}=V_0$$

Here we are interested in motional emf used in generators - arises when wire is moved through a magnetic field $[\oint\vec{E}\cdot\vec{dl}=0]$

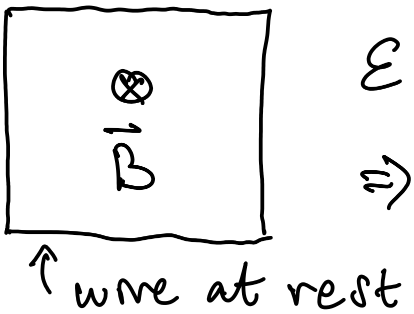


emf: $\mathcal{E}=\oint(\vec{v}\times\vec{B})\cdot\vec{dl}=$

$$=-\oint\vec{B}\cdot\underbrace{(\vec{v}\times\vec{dl})}_{\frac{d}{dt}\vec{dS}}=-\frac{d}{dt}\Phi$$

where the flux: $\Phi\equiv\int\vec{B}\cdot\vec{dS}$ [direction $\vec{dS}\odot$ $\vec{dl}$]

or when the $\vec{B}$-field is changed



$$\mathcal{E}=\oint\vec{E}\cdot\vec{dl}\overset{\text{exp. found}}{=}-\frac{d}{dt}\Phi=-\int\left(\frac{\partial}{\partial t}\vec{B}\right)\cdot\vec{dS}$$

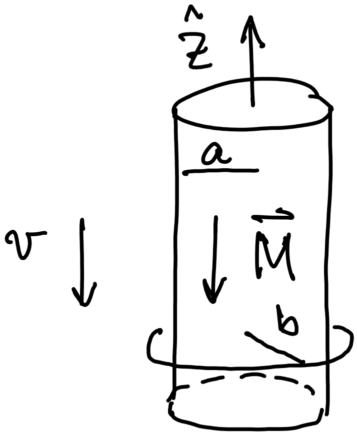
$$\Rightarrow\vec{\nabla}\times\vec{E}=-\frac{\partial}{\partial t}\vec{B}$$ Faraday's law

Lenz's law

note: $\vec{\nabla}\times\vec{E}\neq0 \Rightarrow \vec{E}\neq-\vec{\nabla}V$

$$\vec{B}=\vec{\nabla}\times\vec{A}\Rightarrow\vec{\nabla}\times\left(\vec{E}+\frac{\partial}{\partial t}\vec{A}\right)=0\Rightarrow\vec{E}=-\vec{\nabla}V-\frac{\partial}{\partial t}\vec{A}$$

Ex. 7.5 A long cylindrical magnet (length L, radius a) with uniform magnetization $\vec{M}$ passes through the a circular wire ring (radius b) with velocity v. What is the flux and induced emf?



flux through ring:

$$\Phi_{before} = \Phi_{after} = 0$$

$$\Phi_{during} = \int \vec{B} \cdot d\vec{S} = \mu_0 M \pi a^2$$

$$\left[J_f = 0 \Rightarrow \vec{H} = 0 \Rightarrow \vec{B} = \mu_0 \vec{M} \right]^{?*}$$

time to pass through $t = \frac{L}{v}$

$$\therefore \Phi = \Theta(t - t_0)\left(1 - \Theta\left(t - \left(t_0 + \frac{L}{v}\right)\right)\right) \mu_0 M \pi a^2$$

The emf: $\mathcal{E} = -\frac{d}{dt}\Phi = -\mu_0 M \pi a^2 \left[\delta(t - t_0) - \delta\left(t - \left(t_0 + \frac{L}{v}\right)\right)\right]$

How large are the corresponding current spikes if the wire has conductivity σ?

induced $\vec{E}$-field: $\mathcal{E} = \oint \vec{E} \cdot d\vec{l} = E\, 2\pi b$

Ohm's law: $\vec{J} = \sigma \vec{E}$

$$\Rightarrow \vec{J} = +\frac{1}{2}\sigma \mu_0 M \frac{a^2}{b}\left[\delta(t - t_0) - \delta\left(t - \left(t_0 + \frac{L}{v}\right)\right)\right]\hat{\varphi}$$

↑ direction from Lentz law

alt: $\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}\vec{B}$ (cf. Ampere's law)

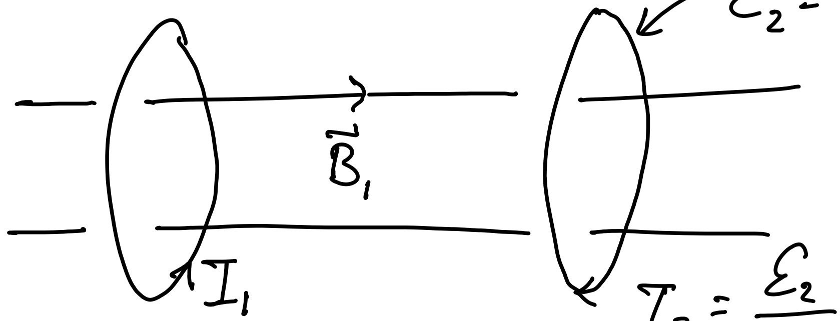
$$\int (\vec{\nabla} \times \vec{E}) \cdot d\vec{S} = -\frac{\partial}{\partial t}\int \vec{B} \cdot d\vec{S}$$

$$\oint \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t}\Phi$$

*also assumes $\vec{B}$ unaffected by change in E

Inductance:

Consider two loops of wire - varying the current in one of them will induce a current in the other one



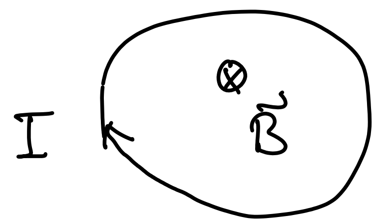
$$\mathcal{E}_2 = -\frac{d}{dt}\Phi_2 = -M\frac{d}{dt}I_1$$

M: mutual inductance (given by geometry)

$$I_2 = \frac{\mathcal{E}_2}{R_2}$$

[Biot-Savart's law $\Rightarrow$ $M = \frac{\mu_0}{4\pi}\oint\oint \frac{d\vec{l}_1 \cdot d\vec{l}_2}{|\vec{r}_1 - \vec{r}_2|}$]

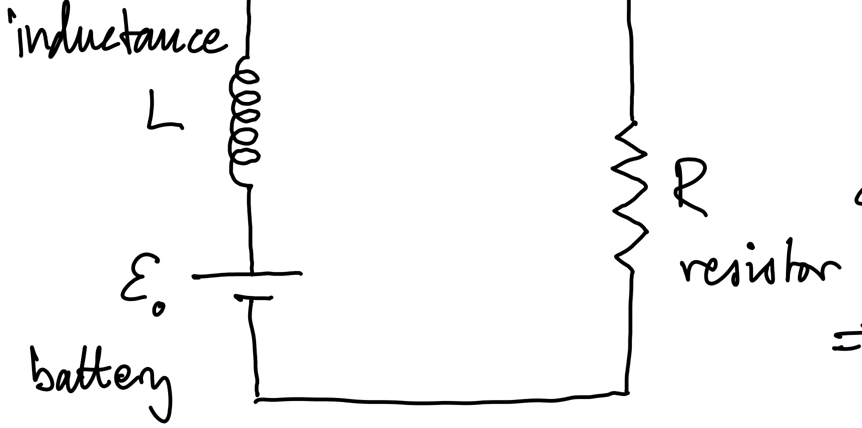
(self) inductance: $\Phi = LI$



$$\mathcal{E} = -L\frac{dI}{dt}$$

L: inductance

Example: consider simple circuit



Ohm's law

$$\mathcal{E}_0 - L\frac{dI}{dt} = RI$$

close circuit at $t=0$

$$\Rightarrow I(t) = \frac{\mathcal{E}_0}{R}\left[1 - e^{-\frac{R}{L}t}\right]$$

takes recoverable energy to build up current

$$\frac{d}{dt} W_m = -\oint \vec{E}\cdot d\vec{I} = -I\mathcal{E} = +I\frac{d}{dt}\Phi = LI\frac{d}{dt}I$$

note: "RI - part" not included in $\mathcal{E}$ - "wasted" as heat

$$\Rightarrow W_m = \frac{1}{2}LI^2 = \frac{1}{2}I\Phi$$

to generalize to arbitrary currents we use

$$\Phi = LI = \int \vec{B}\cdot d\vec{S} = \int (\vec{\nabla}\times\vec{A})\cdot d\vec{S} = \oint \vec{A}\cdot d\vec{l}$$

$$\Rightarrow W_m = \frac{1}{2}I\oint \vec{A}\cdot d\vec{l} = \frac{1}{2}\oint (\vec{A}\cdot\vec{I})\,dl$$

for an arbitrary current density one gets

$$W_m = \frac{1}{2}\int_V (\vec{A}\cdot\vec{J})\,d^3\vec{r} \quad \left(W_e = \frac{1}{2}\int_V V\rho\, d^3\vec{r}\right)$$

in magnetostatic case $\left(\frac{\partial}{\partial t}\vec{E} = 0\right)$ we can use

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$$

to rewrite the energy as

$$W_m = \frac{1}{2\mu_0}\int_V \vec{A}\cdot(\vec{\nabla}\times\vec{B})\,d^3\vec{r} = \quad \left|\begin{array}{l}\vec{A}\cdot(\vec{\nabla}\times\vec{B}) = \\ \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{\nabla}\cdot(\vec{A}\times\vec{B})\end{array}\right|$$

$$= \frac{1}{2\mu_0}\left\{\int_V \vec{B}^2\,d^3\vec{r} - \underbrace{\int_V \vec{\nabla}\cdot(\vec{A}\times\vec{B})\,d^3\vec{r}}_{\oint_S (\vec{A}\times\vec{B})\cdot d\vec{S}}\right\}$$

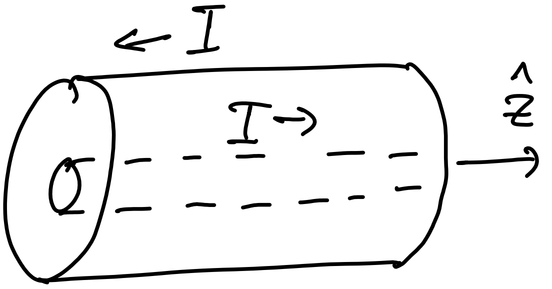
integrate over all of space

$$W_m = \frac{1}{2\mu_0} \int_V \vec{B}^2 \, d^3\vec{r} \qquad \left(W_e = \frac{\varepsilon_0}{2} \int \vec{E}^2 d^3\vec{r} \right)$$

for linear (homogeneous and isotropic) materials this generalises to

$$W_m = \frac{1}{2} \int \vec{B} \cdot \vec{H} \, d^3r \qquad \left(W_e = \frac{1}{2} \int \vec{D} \cdot \vec{E} \, d^3\vec{r} \right)$$

Example: energy stored in long coaxial cable carrying a current I with inner radius a and outer radius b per unit length



Ampere's law: $\vec{B}(a<s<b) = \frac{\mu_0 I}{2\pi s} \hat{\varphi}$

$$\Rightarrow W = \frac{1}{2\mu_0} \int_{\text{cable}} \left(\frac{\mu_0 I}{2\pi s} \right)^2 d^3r =$$

$$= \frac{\mu_0 I^2}{8\pi^2} \int_{\text{cable}} \frac{1}{s^2} \, s\,ds\,d\varphi\,dz = \frac{\mu_0 I^2}{4\pi} \underbrace{\Big[\ln s \Big]_a^b}_{\ln \frac{b}{a}} \underbrace{\int dz}_{l}$$

$$\therefore \frac{W}{l} = \frac{\mu_0 I^2}{4\pi} \ln \frac{b}{a}$$

also: $W = \frac{1}{2} L I^2 \Rightarrow \frac{L}{l} = \frac{\mu_0}{2\pi} \ln \frac{b}{a}$

Maxwells equations (in matter):

Summary:

$$\begin{cases} \vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\rho \\ \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} + \frac{\partial}{\partial t}\vec{B} = 0 \\ \vec{\nabla}\times\vec{B} - \mu_0\varepsilon_0\frac{\partial}{\partial t}\vec{E} = \mu_0\vec{J} \end{cases}$$

note: if we take the divergence of Ampere's law with Maxwells addition $\left(\mu_0\varepsilon_0\frac{\partial}{\partial t}\vec{E}\right)$ we get

$$\underbrace{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B})}_{\equiv 0} - \mu_0\varepsilon_0\underbrace{\vec{\nabla}\cdot\left(\frac{\partial}{\partial t}\vec{E}\right)}_{\frac{\partial}{\partial t}\underbrace{(\vec{\nabla}\cdot\vec{E})}_{\frac{1}{\varepsilon_0}\rho}} = \mu_0\vec{\nabla}\cdot\vec{J}$$

$\therefore \quad \vec{\nabla}\cdot\vec{J} = -\frac{\partial}{\partial t}\rho$, <u>the continuity equation</u>

<u>In matter</u>:

$$\vec{D} = \varepsilon_0\vec{E} + \vec{P}$$

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$$

$$\rho = \rho_f + \rho_b = \rho_f - \vec{\nabla}\cdot\vec{P}$$

$$\vec{J} = \vec{J}_f + \vec{J}_b + \vec{J}_p = \vec{J}_f + \vec{\nabla}\times\vec{M} + \frac{\partial}{\partial t}\vec{P}$$

polarisation current

Cont. eqn: $\vec{\nabla}\cdot\vec{J}_p = -\frac{\partial}{\partial t}\rho_b$

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{M}) = 0$$

$\Rightarrow$ Maxwell's equ's:
$$\begin{cases} \vec{\nabla}\cdot\vec{D} = \rho_f \\ \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} + \frac{\partial}{\partial t}\vec{B} = 0 \\ \vec{\nabla}\times\vec{H} - \underbrace{\frac{\partial}{\partial t}\vec{D}}_{\vec{J}_d} = \vec{J}_f \end{cases}$$

$\vec{J}_d$ displacement current

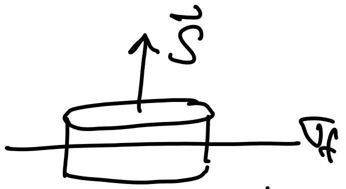
Ex. charging capacitor

linear, isotropic, homogeneous material:

$$\Rightarrow \quad \vec{P} = \varepsilon_0 \chi_e \vec{E}, \quad \vec{M} = \chi_m \vec{H}$$
$$\vec{D} = \underbrace{\varepsilon}_{\varepsilon_0(1+\chi_e)} \vec{E}, \quad \vec{H} = \frac{1}{\underbrace{\mu}_{\mu_0(1+\chi_m)}} \vec{B}$$

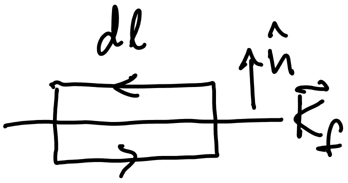
<u>boundary conditions:</u>

Integral form:
$$\begin{cases} \oint_S \vec{D}\cdot d\vec{S} = Q_{f,enc} \\ \oint_S \vec{B}\cdot d\vec{S} = 0 \\ \oint_P \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{S} \\ \oint_P \vec{H}\cdot d\vec{l} = I_{f,enc} + \frac{d}{dt}\int_S \vec{D}\cdot d\vec{S} \end{cases}$$



1 $\quad D_1^\perp - D_2^\perp = \sigma_f$

2 $\quad B_1^\perp - B_2^\perp = 0$



1 $\quad \vec{E}_1^{\parallel} - \vec{E}_2^{\parallel} = 0$

2 $\quad \vec{H}_1^{\parallel} - \vec{H}_2^{\parallel} = \vec{K}_f \times \hat{n}$

[time derivatives continuous $\Rightarrow$ no contrib. as $dS \to 0$]

Energy transport:

Consider a system of charges which are affected by (time-dependent) electromagnetic fields

Work done on charges during dt is

$$dW = \vec{F}\cdot d\vec{l} = q\,(\vec{E} + \vec{v}\times\vec{B})\cdot\vec{v}\,dt = q\,\vec{E}\cdot\vec{v}\,dt$$

for a continuous distribution $\vec{J} = \rho\,\vec{v}\,d^3\vec{r}$ we get

$$\frac{dW}{dt} = \int_V \vec{E}\cdot\vec{J}\,d^3\vec{r}$$

now $\vec{J} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \varepsilon_0\frac{\partial}{\partial t}\vec{E}$

$$\Rightarrow \frac{dW}{dt} = \int_V \Big[\frac{1}{\mu_0}\underbrace{\vec{E}\cdot(\vec{\nabla}\times\vec{B})}_{\vec{B}\cdot\underbrace{(\vec{\nabla}\times\vec{E})}_{-\frac{\partial}{\partial t}\vec{B}} - \vec{\nabla}\cdot(\vec{E}\times\vec{B})} - \varepsilon_0\vec{E}\cdot\frac{\partial}{\partial t}\vec{E}\Big]\,d^3\vec{r}$$

$$= \int_V \Big[-\frac{1}{\mu_0}\underbrace{\Big(\vec{B}\cdot\frac{\partial}{\partial t}\vec{B}\Big)}_{\frac{1}{2}\frac{\partial}{\partial t}\vec{B}^2} - \varepsilon_0\underbrace{\Big(\vec{E}\cdot\frac{\partial}{\partial t}\vec{E}\Big)}_{\frac{1}{2}\frac{\partial}{\partial t}\vec{E}^2} - \vec{\nabla}\cdot\underbrace{\Big(\frac{1}{\mu_0}\vec{E}\times\vec{B}\Big)}_{\equiv\vec{S}}\Big]\,d^3\vec{r}$$

Poynting's vector

$$\therefore \frac{dW}{dt} = \underbrace{-\frac{d}{dt}\int_V \underbrace{\frac{1}{2}\Big(\frac{1}{\mu_0}\vec{B}^2 + \varepsilon_0\vec{E}^2\Big)}_{\text{energy density, } u}\,d^3\vec{r}}_{\text{total energy in } V} - \underbrace{\oint_S \vec{S}\cdot d\vec{S}}_{\text{energy transport out of } V}$$

area

differentially:

$$\vec{J}\cdot\vec{E} + \frac{\partial}{\partial t}u + \vec{\nabla}\cdot\vec{S} = 0$$

work done by $\vec{E}$-field on sources

change in energy density in EM-fields

energy flux out of volume

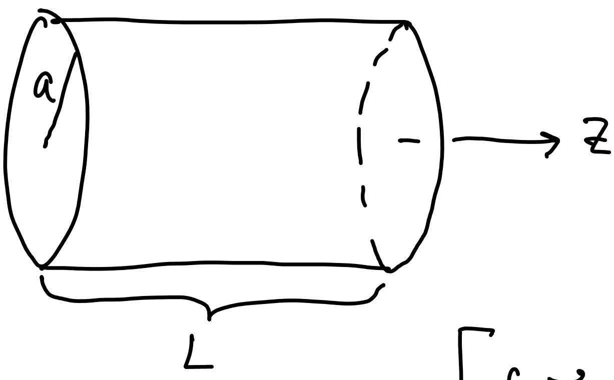
For linear media this generalises to

$$\vec{J}_f\cdot\vec{E} + \frac{\partial}{\partial t}u + \vec{\nabla}\cdot\vec{S} = 0$$

$$u = \frac{1}{2}\left(\vec{D}\cdot\vec{E} + \vec{B}\cdot\vec{H}\right)$$

$$\vec{S} = \vec{E}\times\vec{H}$$

Ex: Joule heating of cylindrical wire (radius a, length L) with uniform electric field $E = V/L$ carrying a total current I.



$$\vec{E} = \frac{V}{L}\hat{z}$$

$$\vec{I} = I\hat{z}$$

Ampere's law: $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$

$$\Rightarrow 2\pi s B = \mu_0 I \frac{s^2}{a^2} \Rightarrow \vec{B} = \frac{\mu_0 I}{2\pi}\frac{s}{a^2}\hat{\varphi}$$

Poynting's vector: $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = -\frac{VIs}{2\pi a^2 L}\hat{s}$

$\Rightarrow$ Power through surface: $P = -\oint \vec{S}\cdot d\vec{S} = VI\frac{s^2}{a^2}$