## Invitation:

Electromagnetism is about solving **Maxwell's equations**

$$\begin{cases} \vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\rho & \text{(Gauss's law)} \\ \vec{\nabla}\cdot\vec{B} = 0 & \\ \vec{\nabla}\times\vec{E} + \frac{\partial}{\partial t}\vec{B} = 0 & \text{(Faraday's law)} \\ \vec{\nabla}\times\vec{B} - \mu_0\varepsilon_0\frac{\partial}{\partial t}\vec{E} = \mu_0\vec{J} & \text{(Ampère's law with Maxwell's correction)} \end{cases}$$

where $\rho$ and $\vec{J}$ fulfill the continuity equation

$$\frac{\partial}{\partial t}\rho = -\vec{\nabla}\cdot\vec{J}$$

together with the Lorentz force law

$$\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$$

which tells us how charges move

Note that in general the fields, $\vec{E}$ and $\vec{B}$ as well as the density $\rho$ and current $\vec{J}$ are functions of both the position $\vec{r}$ and time $t$. By convention we do not write out this dependence.

We will solve Maxwell's eqns using symmetry arguments as well as direct calculations using vector analysis

Chapter 1 in the book by Griffiths contains most of the mathematics we will need

In this course we will follow the tradition and start with considering the case when there is no time-dependence

$$\frac{\partial}{\partial t}\left(\vec{E}, \vec{B}, \rho, \vec{J}\right) = 0$$

and start by exploring <u>electrostatics</u> followed by <u>magnetostatics</u>. Only then will we include the time-dependence.

Finally, before starting we should also make it clear that you have already encountered many of the ideas and concepts of the course in earlier courses (Physics 1) – the difference is that now we will use much more rigor and mathematical tools that will allow us to solve much more complicated problems

# 1. Electrostatics

The relevant Maxwell eqns are

$$\begin{cases} \vec{\nabla} \cdot \vec{E} = \frac{1}{\varepsilon_0} \rho & \left(\frac{\partial}{\partial t} \rho = 0\right) \\ \vec{\nabla} \times \vec{E} = 0 & \left(\frac{\partial}{\partial t} \vec{B} = 0\right) \end{cases}$$

Connection to Coulomb's law?

Force on a testcharge $Q$ from pointcharge $q$ located at $\vec{r}$ and $\vec{r}\,'$ respectively

$$\vec{F} = \frac{1}{4\pi\varepsilon_0} \frac{Qq}{|\vec{r}-\vec{r}\,'|^3} (\vec{r}-\vec{r}\,') \qquad \text{based on exp!}$$

"Convenient" to introduce vector $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$ and define the electric field from $q$

$$\vec{F}(\vec{r}) = Q \underbrace{\frac{1}{4\pi\varepsilon_0} \frac{q}{|\vec{\mathscr{r}}|^3} \vec{\mathscr{r}}}_{\vec{E}(\vec{r})} = Q \frac{1}{4\pi\varepsilon_0} \frac{q}{\mathscr{r}^2} \hat{\mathscr{r}}$$

Generalise to several charges $q_i$ at $\vec{r}_i\,'$

$$\vec{F}(\vec{r}) = Q \sum_i \frac{1}{4\pi\varepsilon_0} \frac{q_i}{|\vec{\mathscr{r}}_i|^3} \vec{\mathscr{r}}_i \qquad (\vec{\mathscr{r}}_i = \vec{r} - \vec{r}_i\,')$$

$$= Q \sum_i \vec{E}_i = Q\, \vec{E}(\vec{r})$$

based on exp!

note: force linear sum – superposition!

and to a continuous distribution $[dq = \rho\, d\tau']$

$$\vec{F} = Q \underbrace{\frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} (\vec{r}-\vec{r}\,')\, d^3\vec{r}\,'}_{\vec{E}(\vec{r})}$$

$[d^3\vec{r}\,' = d\tau']$

where $\rho$ is the density. Three special cases: surface charge $\sigma$, line charge $\lambda$, and point charge $q$. Simple examples:

$$\rho(\vec{r}\,') = \sigma(x', y')\,\delta(z' - z_0)$$

$$\rho(\vec{r}\,') = \lambda(x')\,\delta(y' - y_0)\,\delta(z' - z_0)$$

$$\rho(\vec{r}\,') = q\,\delta^{(3)}(\vec{r}\,' - \vec{r}_0) =$$

$$= q\,\delta(x' - x_0)\,\delta(y' - y_0)\,\delta(z' - z_0)$$

We will make much use of the Dirac $\delta$-fcn

Definition

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a)$$

distribution

example of limiting fcn

$$\delta(x) = \lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi}}\,\frac{1}{\sigma}\,e^{-\frac{x^2}{\sigma^2}}$$

important property

$$\int f(x)\,\delta[g(x)]\,dx = \int f(x)\,\frac{1}{|g'(x)|_{g(x=a)=0}}\,\delta(x-a)\,dx$$

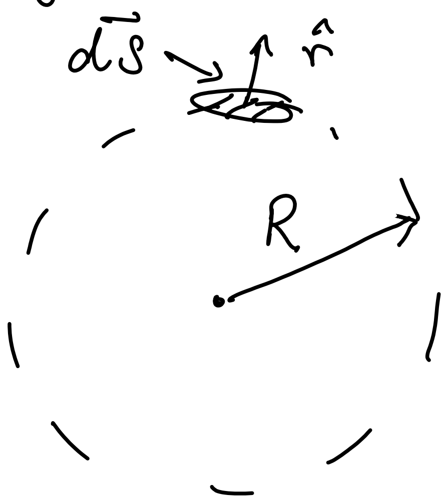
Before proceeding we need one more mathematical result

$$\vec{\nabla}\cdot\left(\frac{\hat{r}}{\vec{r}^{\,2}}\right)$$

1) $r \neq 0$

$$\vec{\nabla}\cdot\left(\frac{\hat{r}}{\vec{r}^{\,2}}\right) = \frac{1}{r^2}\frac{\partial}{\partial r} r^2\left(\frac{1}{r^2}\right) = 0$$

2) applying Gauss' theorem gives




$$\int_V \vec{\nabla}\cdot\left(\frac{\hat{r}}{\vec{r}^{\,2}}\right) d^3\vec{r}$$

$$= \oint_S \frac{\hat{r}}{R^2}\cdot\hat{r}\, R^2 \sin\theta\, d\theta\, d\varphi$$

$$= \int \sin\theta\, d\theta\, d\varphi = 4\pi$$

Only consistent solution is

$$\vec{\nabla}\cdot\left(\frac{\hat{r}}{\vec{r}^{\,2}}\right) = 4\pi\,\delta^{(3)}(\vec{r})$$

Now let's take the divergence of $\vec{E}(\vec{r})$

$$\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot \left( \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}\,')}{(\vec{r}-\vec{r}\,')^2} (\hat{r}-\hat{r}') \, d^3\vec{r}\,' \right)$$

$$= \left[ \vec{\nabla} = \left( \hat{x}\frac{\partial}{\partial x}, \hat{y}\frac{\partial}{\partial y}, \hat{z}\frac{\partial}{\partial z} \right) \text{ only acts on } \vec{r} \right]$$

$$= \frac{1}{4\pi\varepsilon_0} \int \rho(\vec{r}\,') \underbrace{\vec{\nabla} \cdot \left( \frac{\hat{r}-\hat{r}'}{(\vec{r}-\vec{r}\,')^2} \right)}_{4\pi\delta^{(3)}(\vec{r}-\vec{r}\,')} d^3\vec{r}\,' = \frac{1}{\varepsilon_0} \rho(\vec{r}) \; !$$

Gauss's law can also be written on integral form using Gauss's theorem

$$\int_V \vec{\nabla} \cdot \vec{E} \, d^3\vec{r} = \oint_S \vec{E} \cdot d\vec{S}$$

which together with $Q_{enc} = \int_V \rho(\vec{r}) \, d^3\vec{r}$ gives

$$\oint \vec{E} \cdot d\vec{S} = \frac{1}{\varepsilon_0} Q_{enc}$$

Before leaving Coulomb's law we can also verify that $\vec{\nabla} \times \vec{E} = 0$. For simplicity we consider $\vec{E}(\vec{r})$ from a single charge at the origin ($\vec{r}\,' = 0$)

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{q}{\vec{r}^{\,2}} \hat{r}$$

Now take the curl and integrate over a surface $S$ centered at $\vec{r} = 0$

$$\int_S (\vec{\nabla} \times \vec{E}) \cdot d\vec{S} = \oint \vec{E} \cdot d\vec{l}$$

Stokes' theorem

using $d\vec{l} = \hat{r}\, dr + \hat{\theta}\, r\, d\theta + \hat{\varphi}\, r \sin\theta\, d\varphi$
finally gives

$$\oint \vec{E} \cdot d\vec{l} = \frac{1}{4\pi\varepsilon_0} \oint \frac{dr}{\vec{r}^{\,2}} = 0$$

$$\left[ \int \frac{dr}{r^2} = -\frac{1}{r} \right]$$

true for arbitrary $S$ $\Rightarrow$ $\vec{\nabla} \times \vec{E} = 0$

superposition $\Rightarrow$ $\vec{\nabla} \times \vec{E} = 0$ for any static charge distribution

— x —

Connection to field lines?

- start on positive and end on negative charges (or infinity)
- evenly spaced
- in all directions
- field strength given by (3d) density

mathematically these rules follow from

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\varepsilon_0} \rho = \frac{1}{\varepsilon_0} q\, \delta^{(3)}(\vec{r} - \vec{r}\,') \quad \text{for point charge}$$

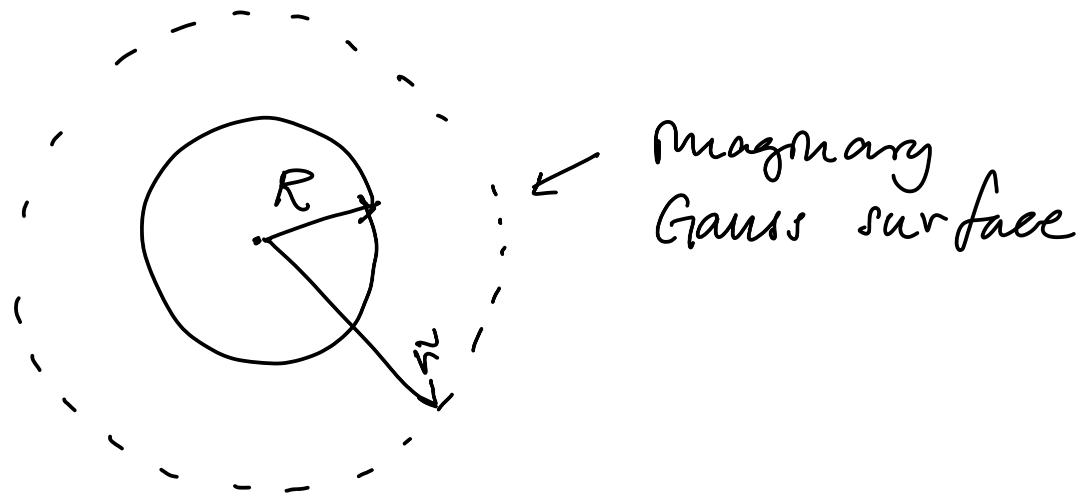
$$\vec{\nabla} \times \vec{E} = 0$$

[cf chapt 1.2.4 and 1.2.5]

Example of (simple) application of Gauss's law:

What is the $\vec{E}$-field outside a uniformly charged solid sphere with radius $R$ and total charge $q$?

Gauss's law: $\oint \vec{E} \cdot d\vec{S} = \frac{1}{\varepsilon_0} Q_{enc}$




1) choose spherical surface centered around the charge $d\vec{S} = \hat{r}\, dS$

2) symmetry dictates that $\vec{E}$ is directed radially outward $\vec{E} = \hat{r}\, E(r)$

3) $Q_{enc} = q$ (surface outside)

$$\Rightarrow \oint \vec{E} \cdot d\vec{S} = E \oint dS = E\, 4\pi r^2 = \frac{1}{\varepsilon_0} q$$

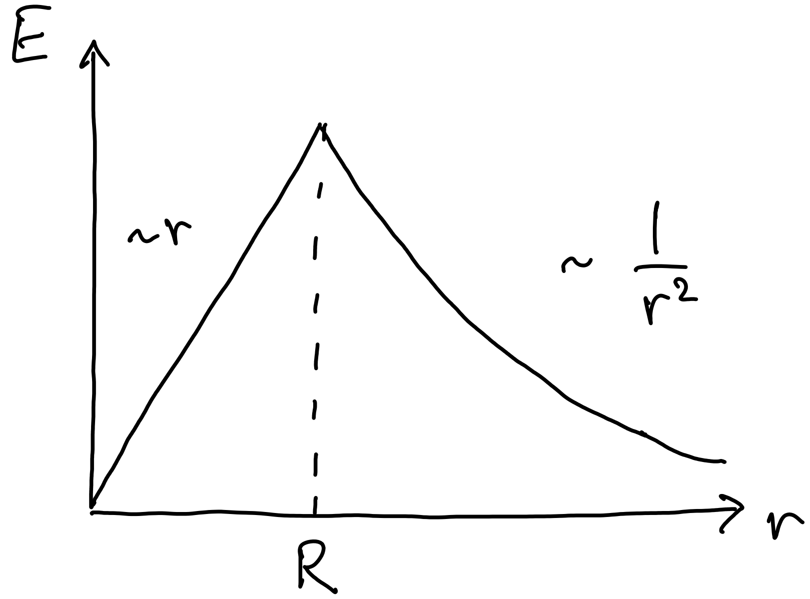
$$\therefore \quad \vec{E}_{outside} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}$$

Can use the same method to calculate the field inside the sphere

3; $Q_{enc} = \int^{r} \rho \, d^3\vec{r}\,' = \rho \int^{r} d^3\vec{r}\,' = \rho \, \frac{4\pi r^3}{3}$

$\left[ \rho = \frac{3q}{4\pi R^3} \right] = q \frac{r^3}{R^3}$

$\therefore \quad \vec{E}_{inside} = \frac{1}{4\pi\varepsilon_0} \frac{qr}{R^3} \hat{r}$



When we want to solve

$$\begin{cases} \vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\rho \\ \vec{\nabla}\times\vec{E} = 0 \end{cases}$$

more generally it is convenient to introduce the "scalar" or electric potential, V

$$\vec{E} = \underset{\uparrow}{-}\vec{\nabla}V$$

convention

or on integral form

$$V(\vec{r}) = -\int_{\vec{O}}^{\vec{r}} \vec{E}\cdot d\vec{l}$$

where $\vec{O}$ defines the boundary cond $V(\vec{O}) = 0$.

This is possible since $\vec{\nabla}\times\vec{E} = 0$ and $\vec{\nabla}\times(\vec{\nabla}f) = 0$ for an arbitrary fcn f.

From the integral formulation it is also easy to see that V obeys the superposition principle (if $\vec{E}$ does)

Inserting $\vec{E} = -\vec{\nabla}V$ into Gauss's law gives

$$\vec{\nabla}\cdot(-\vec{\nabla}V) = \frac{1}{\varepsilon_0}\rho \Rightarrow \vec{\nabla}^2 V = -\frac{1}{\varepsilon_0}\rho$$

called Poisson's equation (we will come back to how to solve this more generally)

Start by finding out V for some charge distribution $\rho$. We know that

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \rho(\vec{r}\,') \underbrace{\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}}_{-\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}\,'|} + \text{const}\right)} d^3\vec{r}\,'$$

acts on $\vec{r}$

$$= -\vec{\nabla} \underbrace{\frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} d^3\vec{r}\,' + \cancel{\text{const}}^{\,0}}_{V(\vec{r})}$$

$$\left[ \because\ V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} d^3\vec{r}\,' + \cancel{\text{const}}^{\,0} \right]$$

for a point charge at $\vec{r}_0\,'$: $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{r}-\vec{r}_0\,'|}$

The arbitrary integration constant is connected to the fact that $V$ is <u>not</u> a physical observable – only differences in $V$ can be measured.

To see this consider a set of charges $q_i$ that produce a total field $\vec{E}$.

Now let's assume we want to add one more charge $Q$ to the system

There is a force on $Q$ from the field

$$\vec{F} = Q\vec{E}$$

to move the charge from $\vec{a}$ to $\vec{b}$ we have to apply an opposite force $\vec{F} = -Q\vec{E}$ and the work done is

$$W = \int \vec{F} \cdot d\vec{l} = -Q \int_{\vec{a}}^{\vec{b}} \vec{E} \cdot d\vec{l} =$$

$$= Q\left[V(\vec{b}) - V(\vec{a})\right]$$

$$\therefore \quad V(\vec{b}) - V(\vec{a}) = \frac{W}{Q}$$

We can also use this to calculate the work (potential energy) needed to assemble a collection of charges from infinity

1) two charges $q_1$ and $q_2$ at $\vec{r}_1$ and $\vec{r}_2$

$$W = q_2 \frac{1}{4\pi\varepsilon_0} \frac{q_1}{|\vec{r}_2 - \vec{r}_1|}$$

[start with $q_1$ and bring in $q_2$ from inf.]

2) three charges

$$W = q_2 \frac{1}{4\pi\varepsilon_0} \frac{q_1}{|\vec{r}_2 - \vec{r}_1|} + q_3 \frac{1}{4\pi\varepsilon_0} \left( \frac{q_1}{|\vec{r}_3 - \vec{r}_1|} + \frac{q_2}{|\vec{r}_3 - \vec{r}_2|} \right)$$

3) n charges

$$W = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n-1} \sum_{j>i}^{n} \frac{q_i q_j}{|\vec{r}_j - \vec{r}_i|}$$

$$= \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \frac{1}{2} \sum_{j \neq i}^{n} \frac{q_i q_j}{|\vec{r}_j - \vec{r}_i|}$$

$$= \frac{1}{2} \sum_{i=1}^{n} q_i \underbrace{\sum_{j \neq i}^{n} \frac{1}{4\pi\varepsilon_0} \frac{q_j}{|\vec{r}_i - \vec{r}_j|}}_{V(\vec{r}_i)\ \text{[from other charges]}} = \frac{1}{2} \sum_{i=1}^{n} q_i V(\vec{r}_i)$$

4) continuous charge distr. [V from all charges]

$$W = \frac{1}{2} \int \rho(\vec{r}\,') \, V(\vec{r}\,') \, d^3\vec{r}\,'$$

<u>note:</u> only valid if $\rho, V$ limited for all $\vec{r}\,'$
so not for point charges
(gives infinite self-energy)

$$= \frac{1}{2} \int \varepsilon_0 \underbrace{\vec{\nabla}' \cdot \vec{E}(\vec{r}\,') \, V(\vec{r}\,')}_{\vec{\nabla}' \cdot \left(\vec{E}(\vec{r}\,') V(\vec{r}\,')\right) - \vec{E}(\vec{r}\,') \cdot \vec{\nabla}' V(\vec{r}\,')} \, d^3\vec{r}\,' =$$

$$= \frac{1}{2} \varepsilon_0 \left[ \underbrace{\oint \overset{\sim \frac{1}{r}}{V(\vec{r}\,')} \, \overset{\sim \frac{1}{r^2}}{\vec{E}(\vec{r}\,')} \cdot \overset{\sim r^2}{d\vec{S}}}_{\to 0 \text{ as } r \to \infty} - \int \vec{E}(\vec{r}\,') \cdot \underbrace{\vec{\nabla}' V(\vec{r}\,')}_{-\vec{E}(\vec{r}\,')} \, d^3\vec{r}\,' \right]$$

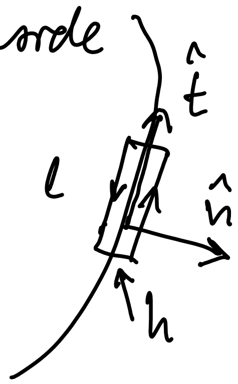
$$\therefore \quad W = \int_{\text{all of space}} \underbrace{\frac{1}{2} \varepsilon_0 \vec{E}^2(\vec{r}\,')}_{u_E(\vec{r}\,'),\ \text{energy density}} \, d^3\vec{r}\,'$$

[again this gives an infinite self-energy of a point charge!]

# Boundary conditions

Consider thin sheet with surface charge $\sigma$

1) inside outside




$\vec{E}_{out} \cdot \hat{t}$ for $\hat{t}$ in plane

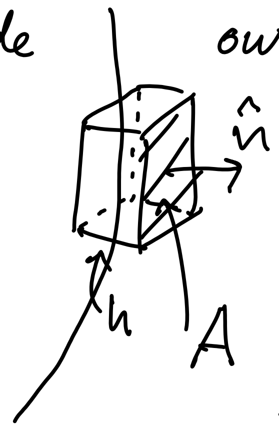
$$\oint \vec{E} \cdot d\vec{l} = E^{\parallel}_{out} \cdot l - E^{\parallel}_{in} \cdot l + O(h)$$

$$= (E^{\parallel}_{out} - E^{\parallel}_{in}) \cdot l = 0$$

$$[\vec{\nabla} \times \vec{E} = 0]$$

$$\Rightarrow \vec{E}^{\parallel}_{out} = \vec{E}^{\parallel}_{in}$$

2) inside outside




$\vec{E}_{out} \cdot \hat{n}$

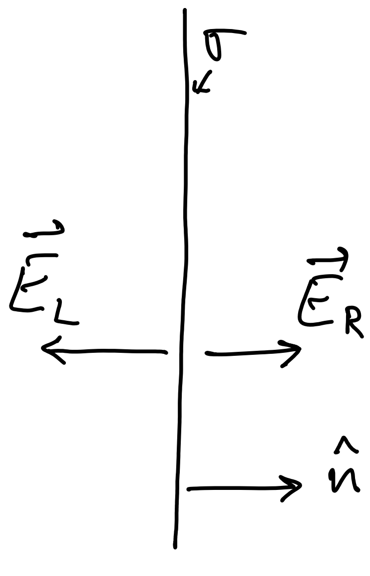
$$\oint \vec{E} \cdot d\vec{S} = E^{\perp}_{out} \cdot A - E^{\perp}_{in} \cdot A$$

$$= \frac{1}{\varepsilon_0} \sigma A$$

$$\Rightarrow E^{\perp}_{out} - E^{\perp}_{in} = \frac{1}{\varepsilon_0} \sigma$$

$$\therefore \quad \vec{E}_{out} - \vec{E}_{in} = \frac{1}{\varepsilon_0} \sigma \hat{n}$$

For an infinite plane with uniform surface charge $\sigma$ this gives




symmetry: $\vec{E}_R = -\vec{E}_L = E(x)\,\hat{n}$

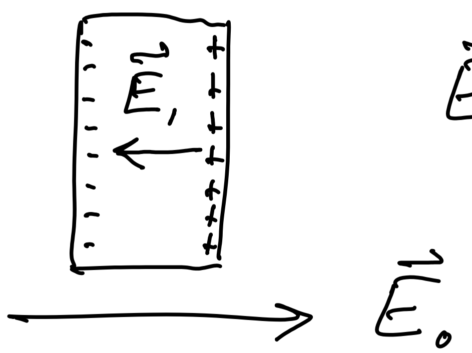
$$\left.\begin{array}{l} \oint \vec{E} \cdot d\vec{S} = 2E(x)A \\ Q_{enc} = A\sigma \end{array}\right\} E(x) = \frac{\sigma}{2\varepsilon_0} \hat{n}$$

independent of distance !

# Conductors

In a perfect conductor the electrons can move completely freely – gives several consequences: valid in electrostatic case!

a) $\vec{E} = 0$ inside – charges move until field vanishes




$\vec{E}_1 + \vec{E}_0 = 0$ inside

b) as a consequence $\rho_{inside} = \varepsilon_0 \vec{\nabla} \cdot \vec{E} = 0$

c) can have non-zero surface charge $\sigma$

d) the potential $V = -\int_{\mathcal{O}}^{\vec{r}} \vec{E} \cdot d\vec{l} = \text{const}$ (equipotential)

e) $\vec{E} \parallel \hat{n}$ just outside conductor

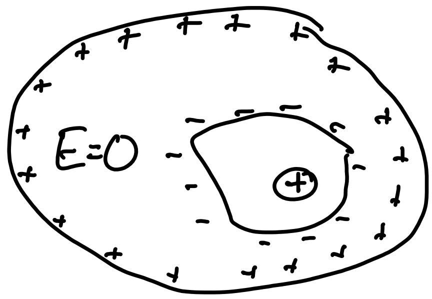
look also at boundary conditions

$$\left.\begin{aligned} E_{in}^{\parallel} &= 0 \Rightarrow E_{out}^{\parallel} = 0 \\ E_{in}^{\perp} &= 0 \Rightarrow E_{out}^{\perp} = \frac{1}{\varepsilon_0}\sigma \end{aligned}\right\} \quad \vec{E}_{out} = \frac{\sigma}{\varepsilon_0}\hat{n}$$

# Induced charges

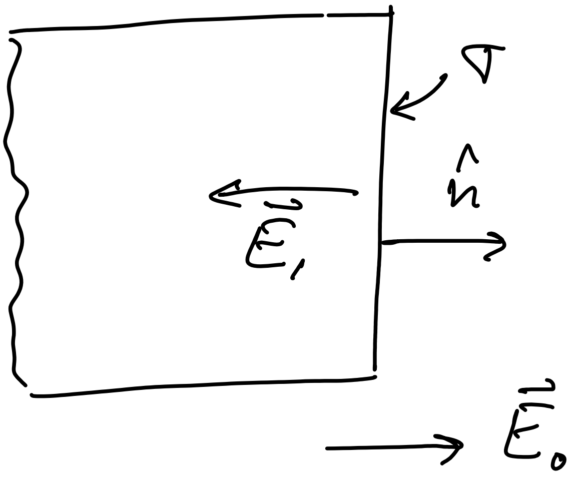
external field polarises conductor to keep $\vec{E} = 0$ inside
$\Rightarrow$ $\rho$ modified in non-trivial way

Ex. uncharged conductor with point-charge $q$ inside cavity




Field outside same as from conductor charged with $q$ - irrespective of shape and position of cavity and position of point charge

# Force on a conductor in external field




general $\quad$ for cond. $\sigma$ is such that

$$\vec{E}_{in} \overset{\text{general}}{=} \vec{E}_1 + \vec{E}_0 = 0$$

$$\vec{E}_{out} = -\vec{E}_1 + \vec{E}_0 = \frac{\sigma}{\varepsilon_0}\hat{n}$$

$$\Rightarrow \vec{E}_0 \underset{\text{general}}{=} \frac{1}{2}\left(\vec{E}_{in} + \vec{E}_{out}\right) \underset{\text{cond.}}{=} \frac{1}{2}\frac{\sigma}{\varepsilon_0}\hat{n}$$

$\Rightarrow$ net force on conductor per surface area

$$\frac{\vec{F}}{A} \underset{\text{general}}{=} \sigma \vec{E}_0 \underset{\text{cond}}{=} \frac{1}{2}\frac{\sigma^2}{\varepsilon_0}\hat{n} = p\,\hat{n}$$

$p$: electrostatic pressure

$$p = \frac{1}{2}\frac{\sigma^2}{\varepsilon_0} = \frac{1}{2}\varepsilon_0 E_{out}^2$$

# Capacitors

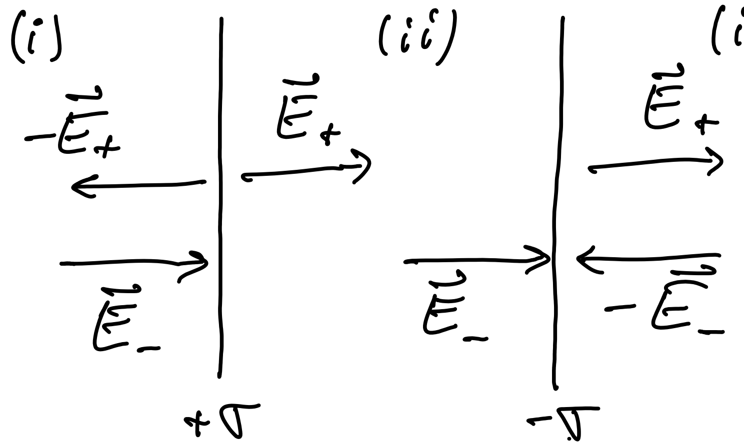
Consider two conductors with charges $\pm Q$ and potential difference $V = V_+ - V_-$ then

$$C \equiv \frac{Q}{V}$$

[the pot. diff. is prop to the charge, "same" charge on both cond]

is the capacitance

Example: two oppositely charged "infinite" planes of perfect conductor with uniform surface charges $\pm\sigma$, $\sigma = \frac{Q}{A}$




$$E_\pm = \frac{\sigma}{2\varepsilon_0}$$

(i) $\vec{E} = \vec{E}_- - \vec{E}_+ = 0$

(ii) $\vec{E} = \vec{E}_- + \vec{E}_+ = \frac{\sigma}{\varepsilon_0}$

(iii) $\vec{E} = -\vec{E}_- + \vec{E}_+ = 0$

The potential difference

$$V = -\int_{(-)}^{(+)} \vec{E} \cdot \vec{dl} = |\vec{E}|\, d = \frac{\sigma d}{\varepsilon_0} = \frac{Q d}{A \varepsilon_0}$$

opposite direction to $\vec{E}$

$$\Rightarrow C = \frac{A\varepsilon_0}{d}$$

Energy stored in capacitor?

Consider a capacitor with potential difference $V'$ and charge $Q' = CV'$

Making an infinitesimal increase of the charge, $Q' \to Q' + dQ'$ the work done is

$$dW = V' dQ' = \frac{1}{C} Q' dQ'$$

integrating from 0 to $Q$ gives

$$W = \int_0^Q \frac{1}{C} Q' dQ' = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} C V^2$$

note: C depends on geometry - is constant when charging a capacitor but V and Q changes (in proportion)