Magnetic fields in matter

A magnetic material in a magnetic field becomes magnetized (and contributes to $\vec{B}$)

paramagnetic: $\vec{M} \propto \vec{B}$ [electron spins line up with $\vec{B}$, $\vec{N} = \vec{m} \times \vec{B}$]

diamagnetic: $\vec{M} \propto -\vec{B}$ [orbital speed of electrons increase cf. induction]

ferromagnetic: $\vec{M} = \text{const} + \ldots$

The magnetization $\vec{M}$ is the magnetic dipole moment per unit volume.

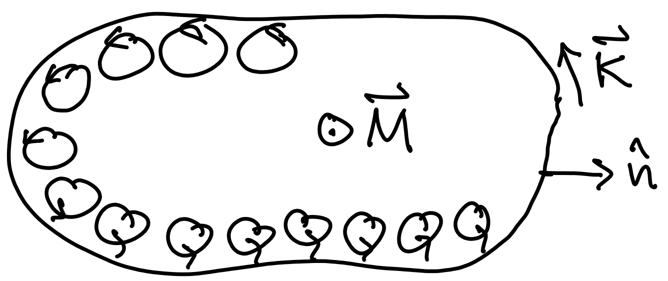
$\Rightarrow$ magnetized object has vector potential

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \vec{M}(\vec{r}') \times \underbrace{\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}}_{\vec{\nabla}' \frac{1}{|\vec{r}-\vec{r}'|}} d^3\vec{r}' =$$

rewrite using int. by parts

$$= \frac{\mu_0}{4\pi} \left[\int \frac{1}{|\vec{r}-\vec{r}'|} \underbrace{(\vec{\nabla}' \times \vec{M}(\vec{r}'))}_{\vec{J}_b(\vec{r}')} d^3\vec{r}' - \underbrace{\int \vec{\nabla}' \times \left(\frac{\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|} \right) d^3\vec{r}'}_{-\oint_S \frac{1}{|\vec{r}-\vec{r}'|} \underbrace{\vec{M}(\vec{r}') \times \hat{n}}_{\vec{K}_b(\vec{r}')} dS'} \right] =$$

$$= \frac{\mu_0}{4\pi} \left[\int \frac{\vec{J}_b(\vec{r}')}{|\vec{r}-\vec{r}'|} d^3\vec{r}' + \oint_S \frac{\vec{K}_b(\vec{r}')}{|\vec{r}-\vec{r}'|} dS' \right]$$



bound surface current

The $\vec{H}$-field:

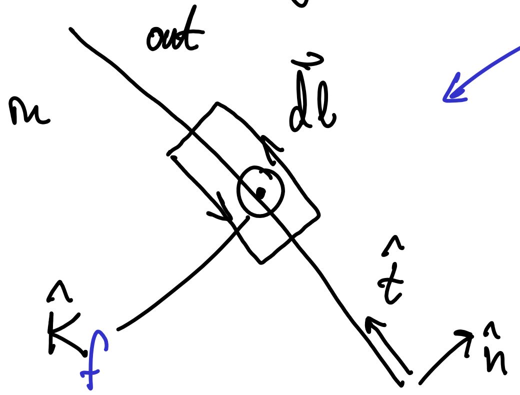
Write total current as $\vec{J} = \vec{J}_b + \vec{J}_f$

$$\Rightarrow \frac{1}{\mu_0}\vec{\nabla}\times\vec{B} = \vec{J}_b + \vec{J}_f = \vec{\nabla}\times\vec{M} + \vec{J}_f$$

$$\therefore \quad \vec{\nabla}\times\underbrace{\left(\frac{1}{\mu_0}\vec{B} - \vec{M}\right)}_{\vec{H}} = \vec{J}_f$$

integral form $\oint \vec{H}\cdot d\vec{l} = I_{f,enc}$

$\Rightarrow$ boundary cond.: $(\vec{H}_{out} - \vec{H}_{in})\cdot\hat{t} = (\vec{K}_f \times \hat{n})\cdot\hat{t}$



as before $\vec{\nabla}\cdot\vec{B} = 0$

$$\Rightarrow \quad \vec{\nabla}\cdot\vec{H} = -\vec{\nabla}\cdot\vec{M} \neq 0$$

$$(\vec{H}_{out} - \vec{H}_{in})\cdot\hat{n} = -(\vec{M}_{out} - \vec{M}_{in})\cdot\hat{n}$$

Linear media (+ isotropic + homogeneous)

Magnetization prop. to $\underline{\underline{\vec{H}}}$-field

magnetic susceptibility

$$\vec{M} = \chi_m \vec{H}$$

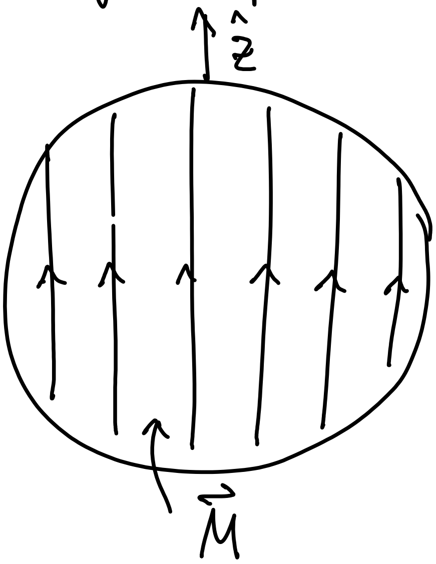
$$\Rightarrow \quad \vec{B} = \mu_0(\vec{H} + \vec{M}) = \underbrace{\mu_0\overbrace{(1+\chi_m)}^{\mu_r}}_{\mu}\vec{H}$$

μ ← permeability

note: $\chi_m < 0$ diamagnetic

$\chi_m > 0$ paramagnetic

Example: Find the magnetic field of a uniformly magnetized sphere with radius R. (or four problems in one)



Choose coordinate system such that

$$\vec{M} = M\hat{z} = M(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})$$

$$\Rightarrow \vec{J}_b = \vec{\nabla} \times \vec{M} = M\frac{1}{r}\left[\frac{\partial}{\partial r}(-r\sin\theta) - \frac{\partial}{\partial\theta}(\cos\theta)\right]\hat{\varphi} = 0$$

$$\vec{K}_b = \vec{M} \times \hat{r} = M\sin\theta\,\hat{\varphi}$$

Want to find $\vec{A}$. By definition

$$\vec{A} = \frac{\mu_0}{4\pi}\int \vec{M}(\vec{r}\,') \times \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\, d^3\vec{r}\,'$$

alt. calc. $\vec{A}$ using $\vec{K}_b$
$$\vec{A} = \frac{\mu_0}{4\pi}\oint \frac{\vec{K}_b(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}dS'$$
(Ex. 5.11)

inside the sphere we have $\vec{M} = \text{const}$

$$\Rightarrow \vec{A} = \frac{\mu_0}{4\pi} M\,\hat{z} \times \int \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\, d^3\vec{r}\,'$$

compare with uniformly polarized sphere

$$V = \frac{1}{4\pi\varepsilon_0} P\,\hat{z} \cdot \int \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\, d^3\vec{r}\,'$$

(which has same solution as metal sphere in otherwise uniform $\vec{E}$-field, except boundary cond) with ($\sigma_b = \vec{P}\cdot\hat{r} = P\cos\theta$ and) solution (sep. of var.)

$$V = \frac{1}{4\pi\varepsilon_0} P\,\hat{z} \cdot \begin{cases} \frac{4\pi}{3}\vec{r}, & r \le R \\ \frac{4\pi}{3}\frac{R^3}{r^3}\vec{r}, & r \ge R \end{cases}$$

As a consequence;

$$\vec{A} = \frac{\mu_0}{4\pi} M \hat{z} \times \begin{cases} \frac{4\pi}{3} \vec{r} & , r \leq R \\ \frac{4\pi}{3} \frac{R^3}{r^3} \vec{r} & , r \geq R \end{cases}$$

$$= \frac{\mu_0}{3} \vec{M} \times \vec{r} \begin{cases} 1 & , r \leq R \\ \frac{R^3}{r^3} & , r \geq R \end{cases}$$

Thus the $\vec{B}$-field is

$$\vec{B}_{in} = \vec{\nabla} \times \vec{A}_{in} = \frac{\mu_0}{3} \vec{\nabla} \times (\vec{M} \times \vec{r}) =$$

$$\left[\vec{M} \times \vec{r} = M r \sin\theta \, \hat{\varphi} , \right.$$

$$\vec{\nabla} \times (r \sin\theta \, \hat{\varphi}) = \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (r \sin^2\theta) \, \hat{r} - \frac{1}{r} \frac{\partial}{\partial r} (r^2 \sin\theta) \, \hat{\theta}$$

$$\left. = 2 \cos\theta \, \hat{r} - 2 \sin\theta \, \hat{\theta} = 2 \hat{z} \right]$$

$$= \frac{2}{3} \mu_0 M \hat{z} = \frac{2}{3} \mu_0 \vec{M}$$

$\vec{B}_{out}$ = field from dipole with $\vec{m} = \frac{4\pi R^3}{3} \vec{M}$

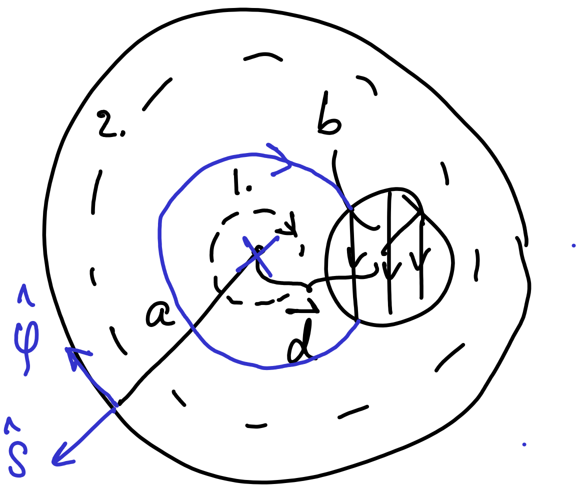
$\vec{H}_{in} = \frac{1}{\mu_0} \vec{B}_{in} - \vec{M} = -\frac{1}{3} \vec{M}$, $\vec{H}_{out} = \frac{1}{\mu_0} \vec{B}_{out}$

sketch $\vec{M}$, $\vec{B}$, $\vec{H}$

Finally, the surface current $\vec{K}_b = M \sin\theta \, \hat{\varphi}$, gives the solution to the $\vec{B}$-field from a spherical shell, with radius R and uniform surface charge σ, spinning at angular velocity ω by the identification $M = \omega \sigma R$

$$\left(\vec{K} = \sigma \vec{v} = \sigma \omega R \sin\theta \, \hat{\varphi} \right)$$

Consider a cylindrical conductor of radius a with a hole of radius b bored parallel to the axis at a distance d. The current density is uniform in the remaining cylinder and parallel to the axis. Use Ampere's law to find out the $\vec{B}$-field.



Ampere's law

$$\oint \vec{B}\cdot d\ell = \mu_0 \int \vec{J}\cdot d\vec{S}$$

Current density

$$\vec{J} = \frac{I}{\pi(a^2-b^2)}\,\hat{z} \quad \text{(into board)}$$

$$1: \quad B_\varphi 2\pi s = \mu_0 \pi s^2 \frac{I}{\pi(a^2-b^2)} \Rightarrow \vec{B} = \mu_0 \frac{s I \hat{\varphi}}{2\pi(a^2-b^2)} = \mu_0 \frac{I \hat{z}\times\vec{r}}{2\pi(a^2-b^2)}$$

$$2: \quad B_\varphi 2\pi s = \mu_0 \pi (s^2-b^2)\frac{I}{\pi(a^2-b^2)} \Rightarrow B_\varphi = \mu_0 \frac{I}{2\pi s}\,\frac{s^2-b^2}{a^2-b^2}$$

3: inside the hole we use superposition

$$\vec{B} = \vec{B}_{\text{without hole}} + \vec{B}_{\text{current in opposite dir. in hole}}$$

$$\Rightarrow \vec{B} = \mu_0 \frac{I\hat{z}\times\vec{r}}{2\pi(a^2-b^2)} + \mu_0 \frac{-I\hat{z}}{2\pi(a^2-b^2)}\times(\vec{r}-\vec{d}) = \mu_0 \frac{I\,\hat{z}\times\vec{d}}{2\pi(a^2-b^2)}$$

$= \text{const}\,!$

$$\hat{\varphi} = -\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y}$$

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$\hat{z}\times\vec{r} = x\hat{y} - y\hat{x} = s\cos\varphi\,\hat{y} - s\sin\varphi\,\hat{x} = s\hat{\varphi}$$

Uniformly magnetized sphere , $\vec{M}$

$$\vec{B}_{in} = \frac{2}{3}\mu_0 \vec{M} \quad , \quad \vec{B}_{out} = \vec{B}_{dipole}$$

$$\vec{H}_{in} = -\frac{1}{3}\vec{M} \quad , \quad \vec{H}_{out} = \frac{1}{\mu_0}\vec{B}_{out}$$