Magnetostatics

Electrostatics: stationary charges $\Rightarrow \vec{E}$

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\rho \qquad \text{(Gauss' law)}$$

$$\vec{\nabla}\times\vec{E} = 0$$

moving charges (currents) $\Rightarrow$ magnetic fields

Magnetostatics: steady currents $\Rightarrow \vec{B}$

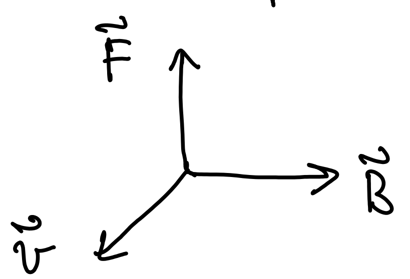
$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{B} = \mu_0 \vec{J} \qquad \text{(Ampere's law)}$$

Equivalent of Coulomb's law? [Biot-Savart, first currents]

Lorentz force on point charge moving in B-field:

$$\vec{F} = q\,\vec{v}\times\vec{B} \qquad \text{(exp. observation)}$$

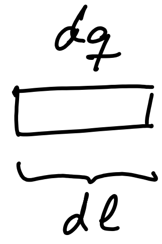


[used HEP exp to measure $\vec{p} = m\vec{v}$]

no work! $dW = \vec{F}\cdot d\vec{l} = q(\vec{v}\times\vec{B})\cdot\vec{v}\,dt = 0$

From now on we will talk about currents instead of point charges (continuous flow)

Line-current:



$$\vec{I} = \lambda\vec{v}$$

↖ line charge dens. $dq = \lambda\, dl$

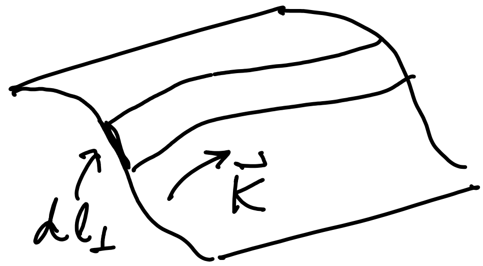
$$d\vec{F} = \vec{v} \times \vec{B}\, dq = \vec{v} \times \vec{B} \lambda\, dl = \vec{I} \times \vec{B}\, dl$$

$$\Rightarrow \vec{F} = \int \vec{I} \times \vec{B}\, dl$$

Surface-current:

$$\vec{K} \equiv \frac{d\vec{I}}{dl_\perp} = \sigma \vec{v}$$

surface charge dens

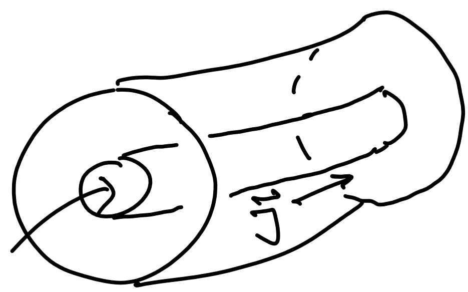


$$\vec{F} = \int \vec{K} \times \vec{B}\, dS = \int \sigma \vec{v} \times \vec{B}\, dS$$

Volume-current:

$$\vec{J} \equiv \frac{d\vec{I}}{da_\perp} = \rho \vec{v}$$

volume charge dens.



$$\vec{F} = \int \vec{J} \times \vec{B}\, d^3\vec{r} = \int \rho \vec{v} \times \vec{B}\, d^3\vec{r}$$

Ex: What is $|\vec{J}|$ on a wire of circular cross-section with radius carrying a uniformly distributed current I

$$J = \frac{I}{\pi a^2}$$

What is I if $J = k\, s$ ← radial distance

$$I = \int J\, da_\perp = \int_0^a ks\, s\, ds\, d\varphi = 2\pi k \frac{a^3}{3}$$

↑ cylindrical coordinates!

Continuity equ:

Electric charge cannot be created or destroyed – charge is conserved

$$\Rightarrow \quad \underbrace{\oint \vec{J} \cdot d\vec{S}}_{\int_V \vec{\nabla} \cdot \vec{J} \, d^3\vec{r}} = -\frac{\partial}{\partial t} \int_V \rho \, d^3\vec{r}$$

$$\therefore \quad \vec{\nabla} \cdot \vec{J} = -\frac{\partial}{\partial t} \rho \quad \left(\partial_\mu j^\mu = 0 \, , \, j^\mu = (c\rho, \vec{J}) \right)$$

So, what does $\vec{B}$ look like?

Biot-Savart law (steady currents, based on exp.)

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \vec{I}(\vec{r}\,') \times \frac{(\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} dl' \quad \text{(line)}$$

$$\vec{K}(\vec{r}\,') \qquad d\vec{S}\,' \quad \text{(surface)}$$

$$\vec{J}(\vec{r}\,') \qquad d^3\vec{r}\,' \quad \text{(volume)}$$

permeability of free space $\mu_0 = 4\pi \times 10^{-7} \frac{N}{A^2}$.

Can in principle be used together with Lorentz force law to solve problems, but there are more powerful ways based on

$$\begin{cases} \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \end{cases}$$

So let's derive these equ's!

Start from

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \vec{J}(\vec{r}') \times \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} d^3\vec{r}'$$

(i) use $\vec{\nabla} \frac{1}{|\vec{r}-\vec{r}'|} = - \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$ to write

$[\vec{\nabla} \times (f\vec{a}) = f(\vec{\nabla}\times\vec{a}) - \vec{a}\times\vec{\nabla}f, \ \vec{\nabla}\times\vec{J}(\vec{r}')=0]$

$$\Rightarrow \vec{B}(\vec{r}) = \vec{\nabla} \times \underbrace{\left(\frac{\mu_0}{4\pi} \int \vec{J}(\vec{r}') \frac{1}{|\vec{r}-\vec{r}'|} d^3\vec{r}' \right)}_{\vec{A}\text{ vector pot.}} \qquad (*)$$

(ii) $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) \equiv 0 \quad \Rightarrow \quad \vec{\nabla}\cdot\vec{B} = 0$

no magnetic monopoles!

integral form $\oint \vec{B}(\vec{r}) \cdot d\vec{S} = 0$

[field lines - closed loops]

(iii) take curl of $\vec{B}$ $(*)$

$[\vec{\nabla}\times(\vec{\nabla}\times\vec{a}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{a}) - \vec{\nabla}^2\vec{a}$

$\vec{\nabla}\cdot(f\vec{a}) = f\vec{\nabla}\cdot\vec{a} + \vec{a}\cdot\vec{\nabla}f]$

$\Rightarrow \vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}$ [$\vec{J}$ function of $\vec{r}'$ not $\vec{r}$]

$$\left\{ \vec{\nabla} \int \vec{J}(\vec{r}') \cdot \underbrace{\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}'|}}_{-\vec{\nabla}'\frac{1}{|\vec{r}-\vec{r}'|}} d^3\vec{r}' - \int \vec{J}(\vec{r}') \underbrace{\vec{\nabla}^2 \frac{1}{|\vec{r}-\vec{r}'|}}_{-4\pi\delta^{(3)}(\vec{r}-\vec{r}')} d^3\vec{r}' \right\}$$

use partial int. on first term

$$\int \vec{J}(\vec{r}')\cdot\vec{\nabla}'\frac{1}{|\vec{r}-\vec{r}'|} d^3\vec{r}' = \int \left[\vec{\nabla}'\cdot\left(\frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right) - \frac{\vec{\nabla}'\cdot\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|} \right] d^3\vec{r}'$$

and

$$\int \vec{\nabla}' \cdot \left(\frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|} \right) d^3\vec{r}' = \oint_S \frac{\overbrace{\vec{J}(\vec{r}')}^{0 \text{ on boundary}} \cdot d\vec{S}}{|\vec{r}-\vec{r}'|} = 0$$

$$\Rightarrow \vec{\nabla} \times \vec{B} = \frac{\mu_0}{4\pi} \left[\vec{\nabla} \int \frac{\vec{\nabla}' \cdot \vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|} d^3\vec{r}' + 4\pi \vec{J}(\vec{r}) \right] =$$

[continuity eqn: $\vec{\nabla} \cdot \vec{J} = -\frac{\partial}{\partial t} \rho$]

$$= \mu_0 \vec{J}(\vec{r}) + \underbrace{\frac{\mu_0}{4\pi} \vec{\nabla} \left(-\frac{\partial}{\partial t} \underbrace{\int \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} d^3\vec{r}}_{4\pi\varepsilon_0 V(\vec{r})} \right)}_{\mu_0 \varepsilon_0 \frac{\partial}{\partial t} \vec{E}}$$

$$\therefore \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \underbrace{\mu_0 \varepsilon_0 \frac{\partial}{\partial t} \vec{E}}_{\substack{=0 \text{ for} \\ \text{steady currents}}}$$ <u>Ampere's law</u>

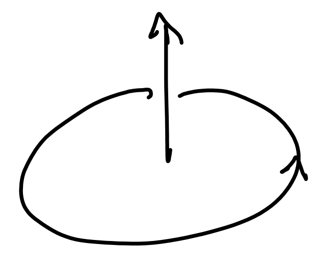
Integral form:

$$\int (\vec{\nabla} \times \vec{B}) \cdot d\vec{S} = \mu_0 \underbrace{\int \vec{J} \cdot d\vec{S}}_{I_{enc}}$$

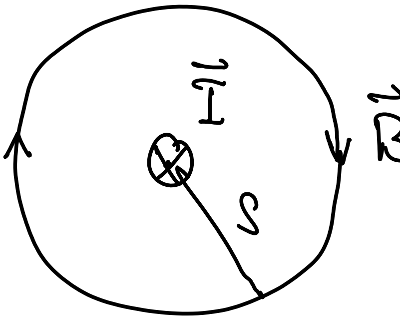
[Stokes' theorem] [through surface]

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

positive direction defined by "right hand rule"



Example: magnetic field a distance s from an infinite straight wire carrying a steady current I



$\vec{I}$ into page

$\vec{B}$ by right-hand rule

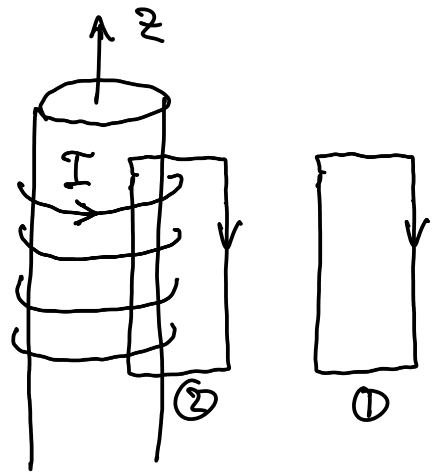
Ampere's law

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 I_{enc}$$

Cylindrical coordinates, $\vec{dl} = \hat{\varphi}\, s\, d\varphi$, $\vec{I} = I\hat{z}$

$$B \cdot 2\pi s = \mu_0 I \Rightarrow \vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\varphi}$$

Example: infinite solenoid with n turns per unit length



$$\oint \vec{B} \cdot \vec{dl} = \mu_0 I_{enc}$$

symmetry $\Rightarrow \vec{B} \propto \hat{z}$

$\vec{B} \to 0$ as $s \to \infty$

1 $\Rightarrow \vec{B}_{out} = \text{const} = 0$

2 $\Rightarrow \vec{B}_{in} = \mu_0 n I \hat{z}$

Ampere's law also useful for infinite planes and toroids (cf ATLAS exp.)

<u>Vector potential</u>:

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|} d^3\tilde{r}'$$

Is this unique?

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

can add $\vec{\nabla}\lambda$ to $\vec{A}$ since $\vec{\nabla} \times (\vec{\nabla}\lambda) \equiv 0$
(cf. const can be added to V)

ambiguity in $\vec{A}$ called gauge freedom

in magnetostatics it is convenient to choose λ such that

$$\vec{\nabla} \cdot \vec{A} = 0 \qquad \underline{\text{Coulomb gauge}}$$

$$\Rightarrow \vec{\nabla} \times \vec{B} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}\underbrace{(\vec{\nabla} \cdot \vec{A})}_{=0} - \vec{\nabla}^2 \vec{A} = \mu_0 \vec{J}$$

(cartesian) components of $\vec{A}$ fulfill Poisson's equ with solution

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|} d^3\tilde{r}$$

if $\vec{\nabla} \cdot \vec{A} \neq 0$ then we add $\vec{\nabla}\lambda$ so that

$$\vec{A}' = \vec{A} + \vec{\nabla}\lambda$$

$$\vec{\nabla} \cdot \vec{A}' = 0 \quad \Rightarrow \quad \vec{\nabla}^2 \lambda = -\vec{\nabla} \cdot \vec{A}$$

sol'n $\lambda = \frac{1}{4\pi} \int \frac{\vec{\nabla} \cdot \vec{A}}{|\vec{r}-\vec{r}'|} d^3\tilde{r}'$ if $\vec{\nabla} \cdot \vec{A} \to 0$ as $r \to \infty$

4-vector potential

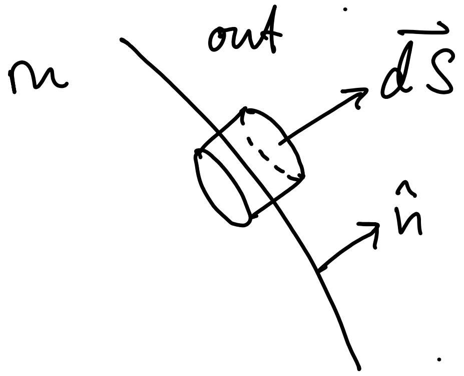
We can combine V and $\vec{A}$ into 4-vector

$$A^\mu = \left(\frac{1}{c}V, \vec{A}\right)$$

Lorentz gauge

$$\partial_\mu A^\mu = 0 \quad \Leftrightarrow \quad \frac{1}{c^2}\frac{\partial}{\partial t}V + \vec{\nabla}\cdot\vec{A} = 0$$

Boundary conditions for $\vec{B}$-field:

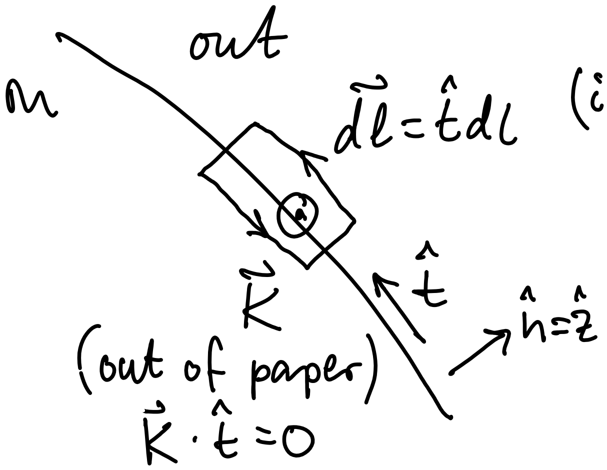


(i) $\vec{\nabla} \cdot \vec{B} = 0$

$$\Rightarrow \oint \vec{B} \cdot d\vec{S} = 0$$

$$\Rightarrow \vec{B}_{out} \cdot \hat{n} - \vec{B}_{in} \cdot \hat{n} = 0$$

$$\therefore B^{\perp}_{out} = B^{\perp}_{in}$$



(ii) $\vec{\nabla} \times \vec{B} = \mu_0 \vec{K}\, \delta(z - z_0)$

$$\int (\vec{\nabla} \times \vec{B}) \cdot d\vec{S} = \mu_0 \int \vec{K}\, \delta(z - z_0) \cdot d\vec{S}$$

$$d\vec{S} \parallel \vec{K}$$

$$\oint \vec{B} \cdot d\vec{l} = \int \mu_0 K\, dl$$

$$\vec{B}_{out} \cdot \hat{t} - \vec{B}_{in} \cdot \hat{t} = \mu_0 K$$

$$\therefore B^{\parallel}_{out} - B^{\parallel}_{in} = \mu_0 K$$

All in all we have

$$\vec{B}_{out} - \vec{B}_{in} = \mu_0 (\vec{K} \times \hat{n})$$

Vector potential: (in Coulomb gauge)

$$\vec{A}_{out} = \vec{A}_{in}$$

$$\frac{\partial}{\partial n} \vec{A}_{out} - \frac{\partial}{\partial n} \vec{A}_{in} = -\mu_0 \vec{K}$$

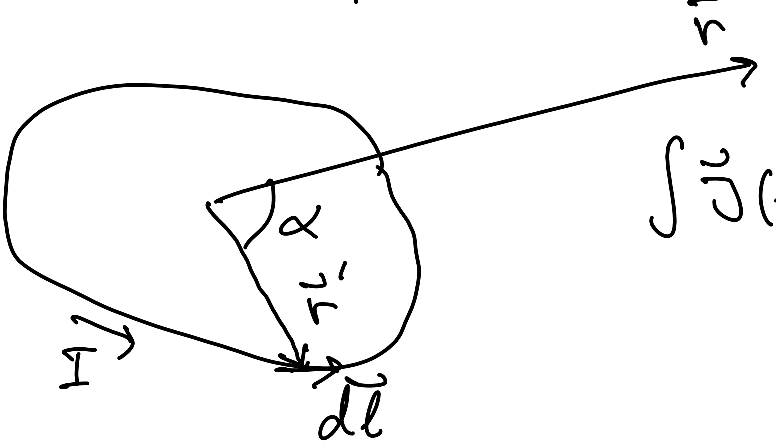
Multipole expansion of $\vec{A}$

Works same way as for V

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d^3\vec{r}\,' =$$

$$= \frac{\mu_0}{4\pi} \int \vec{J}(\vec{r}\,') \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n P_n(\cos\alpha)\, d^3\vec{r}\,'$$

Restrict to case when $\vec{J}(\vec{r}\,')$ describes a current loop



$$\int \vec{J}(\vec{r}\,')\, d^3\vec{r}\,' = I \oint d\vec{l}$$

$$= \frac{\mu_0}{4\pi} I \underbrace{\sum_n \frac{1}{r^{n+1}} \oint (r')^n P_n(\cos\alpha)\, d\vec{l}}$$

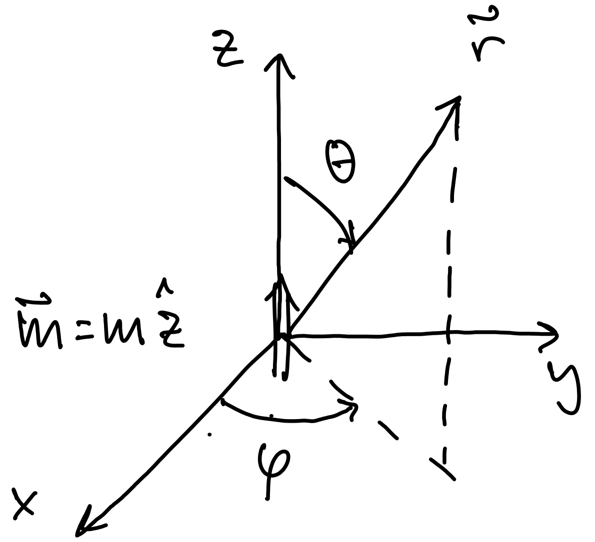
$$\underbrace{\frac{1}{r}\oint d\vec{l}}_{=0} + \underbrace{\frac{1}{r^2}\oint r'\cos\alpha\, d\vec{l}}_{\frac{1}{r^3}\underbrace{\oint \vec{r}\cdot\vec{r}\,' d\vec{l}}_{-\vec{r}\times\int d\vec{S}'}} + \dots$$

[1-page...]

$$= \frac{\mu_0}{4\pi} \frac{\vec{m}\times\vec{r}}{r^3} + \dots \quad , \vec{m} = I\int d\vec{S}$$

magnetic dipole moment

magnetic field from dipole $\left[\vec{m}\times\hat{r} = m\sin\theta\,\hat{\varphi}\right]$



$$\vec{A}_{dip} = \frac{\mu_0}{4\pi}\,\frac{m\sin\theta}{r^2}\,\hat{\varphi}$$

$$\vec{B}_{dip} = \vec{\nabla}\times\vec{A}_{dip} =$$

$$= \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

$$\left(\mu_0\vec{m} \leftrightarrow \frac{1}{\varepsilon_0}\vec{p}\right)$$

For a general current density $\vec{J}$ we have

$$\vec{m} = \frac{1}{2}\int \vec{r}\times\vec{J}\,d^3\vec{r}$$

(see problem 3.38)

Stokes theorem

$$\int_{S'} (\vec{\nabla}' \times \vec{v}) \cdot d\vec{S}' = \oint \vec{v} \cdot d\vec{l}'$$

use $\vec{v} = \vec{c}\,(\vec{r} \cdot \vec{r}')$

$$\vec{\nabla}' \times (\vec{c}\,(\vec{r}\cdot\vec{r}')) = \vec{r}\cdot\vec{r}' \underbrace{(\vec{\nabla}' \times \vec{c})}_{=0} - \vec{c} \times (\vec{\nabla}'(\vec{r}\cdot\vec{r}'))$$

$$= -\vec{c} \times \left(\vec{r} \times (\vec{\nabla}' \times \vec{r}') + \vec{r}' \times (\vec{\nabla}' \times \vec{r}) + (\vec{r}\cdot\vec{\nabla}')\vec{r}' + (\vec{r}'\cdot\vec{\nabla}')\vec{r} \right)$$

$$= -\vec{c} \times (\vec{r}\cdot\vec{\nabla}')\vec{r}' = -\vec{c} \times \vec{r} = \vec{r} \times \vec{c}$$

$$\Rightarrow \int_{S'} (\vec{\nabla}' \times (\vec{c}(\vec{r}\cdot\vec{r}'))) \cdot d\vec{S}' = \int_{S'} (\vec{r} \times \vec{c}) \cdot d\vec{S}' =$$

$$= (\vec{r} \times \vec{c}) \cdot \int d\vec{S}' = \vec{c} \cdot \left(\int d\vec{S}' \times \vec{r} \right) = -\vec{c} \cdot \left(\vec{r} \times \int d\vec{S}' \right)$$

$$\oint \vec{v} \cdot d\vec{l}' = \vec{c} \cdot \oint (\vec{r}\cdot\vec{r}')\, d\vec{l}'$$

$$\Rightarrow -\vec{c} \cdot \left(\vec{r} \times \int d\vec{S}' \right) = \vec{c} \cdot \oint (\vec{r}\cdot\vec{r}')\, d\vec{l}'$$

$\vec{c}$ arbitrary

$$\therefore \quad -\vec{r} \times \int d\vec{S}' = \oint (\vec{r}\cdot\vec{r}')\, d\vec{l}'$$