Calculating the electrostatic potential:

We have seen that the scalar potential can be used to calculate the electric field from a charge distribution

$$V = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} \, d^3\vec{r}\,'$$

problems:

1) integral may be (too) difficult
2) ρ may not be known in detail (conductors)

Three general methods

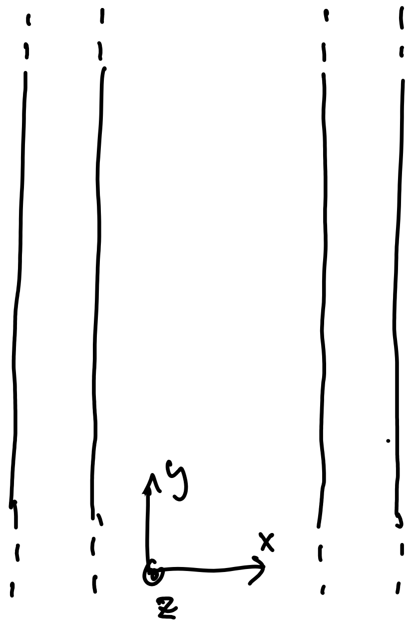
1) solving Poisson's eqn $\nabla^2 V = -\frac{1}{\varepsilon_0}\rho$
2) method of images
3) multipole expansion

Start by looking at Laplace eqn $\nabla^2 V = 0$

- true outside localized charge distr.
- to solve it we will need to know the boundary conditions

Simple 1-d example: (capacitor)

a) two "infinite" parallel conducting plates with potential difference V_0 a distance d apart



$V=0$ $\quad$ $V=V_0$

put one plate at $V=0$ (ground)

$$\frac{d^2}{dx^2}V = 0 \Rightarrow V = ax + b$$

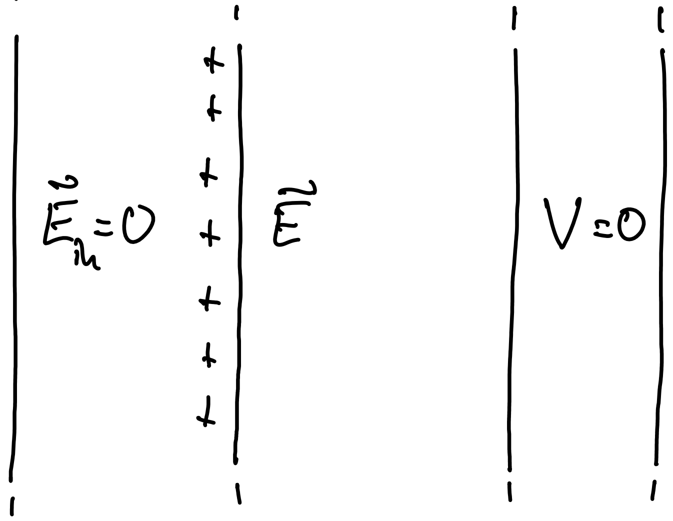
define $x=0$ at $V=0$ plate

boundary cond $\Rightarrow$

$$b = 0 \,, \quad a = \frac{V_0}{d}$$

$$\therefore V = \frac{V_0}{d}x \Rightarrow \vec{E} = -\frac{V_0}{d}\hat{x}$$

b) assume that we instead know the surface charge σ on one of the plates



- $\vec{E}_{in} = 0 \Rightarrow \vec{E}(x=0) = \frac{\sigma}{\varepsilon_0}\hat{x}$

$$\Rightarrow \frac{\partial V}{\partial x}(x=0) = -\frac{\sigma}{\varepsilon_0}$$

$$\Rightarrow a = -\frac{\sigma}{\varepsilon_0}$$

- $V(x=d) = 0$

$$\therefore V = \frac{\sigma}{\varepsilon_0}(d-x)$$

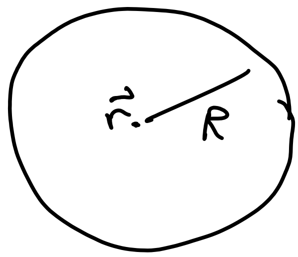
Characteristics of solutions to Laplace eq'n

$$\vec{\nabla}^2 V = 0$$

in some volume with a boundary

1) the potential in one point $\vec{r}$ is given by the average on a spherical surface surrounding it

$$V(\vec{r}) = \frac{1}{4\pi R^2} \oint V \, dS$$



[proof. - see book. Essentially it shows that the average pot. on a spherical surface from a point charge outside is the same as the pot. at the center of the sphere]

$\Rightarrow$ 2) V has no local minima or maxima except on the boundary

$\Rightarrow$ 3) V is smooth and continuous inside volume

$\Rightarrow$ 4) V is uniquely determined by boundary conditions (V or $\frac{\partial}{\partial n} V$, $\hat{n}$ normal to surface) if ρ is known in region of interest

proof: suppose two different sol'ns V_1, V_2
consider $V_3 = V_1 - V_2$

know: $\left.\begin{array}{l} \nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 \overset{\text{on boundary}}{=} 0 \\ V_3 = 0 \text{ on boundary} \end{array}\right\}$

(1-3) $\Rightarrow V_3 \equiv 0 \quad \therefore \quad V_1 = V_2$

called <u>Uniqueness theorem</u>

Method of images:

Powerful tool to solve problems with conductors based on uniqueness theorem
– leave for math. meth. of physics

Solving $\nabla^2 V = 0$ by separation of variables

Usefull when we know the potential or surface charge on the boundary of a region

Ansatz: $V(x,y,z) = X(x)\, Y(y)\, Z(z)$

if we can find a sol'n that satisfies the boundary conditions we are done

Important simplification

$$\frac{\partial^2}{\partial x^2} V = \underset{\uparrow\ \text{total derivative}}{\left[\frac{d^2}{dx^2} X(x)\right]} \underbrace{Y(y)\, Z(z)}_{\text{const. wrt } x}$$

$$\Rightarrow \nabla^2 V = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) V =$$

$$= \left(\frac{d^2}{dx^2} X\right) Y Z + X \left(\frac{d^2}{dy^2} Y\right) Z + X Y \left(\frac{d^2}{dz^2} Z\right)$$

dividing by $V = XYZ$ gives

$$\frac{1}{X}\frac{d^2}{dx^2} X + \frac{1}{Y}\frac{d^2}{dy^2} Y + \frac{1}{Z}\frac{d^2}{dz^2} Z = 0$$

sum of ordinary differential equ's !

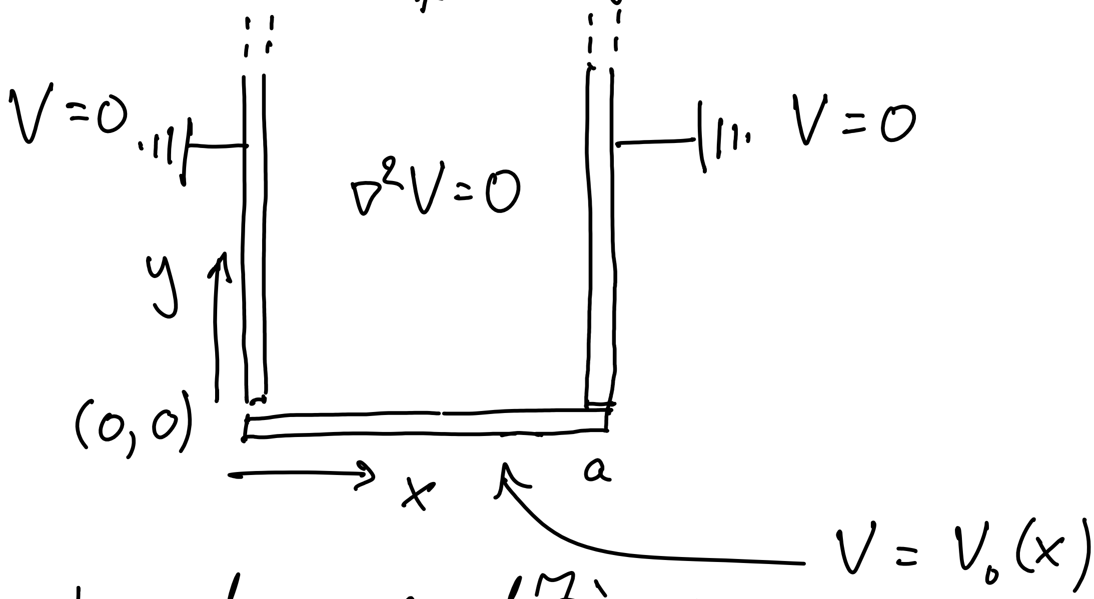
only possible if

$$\frac{1}{X}\frac{d^2}{dx^2} X = C_1, \quad \frac{1}{Y}\frac{d^2}{dy^2} Y = C_2, \quad \frac{1}{Z}\frac{d^2}{dz^2} Z = C_3$$

const.

with $C_1 + C_2 + C_3 = 0$

Example on 2-d Metallic square gutter (cf. example 3.3)



boundary conditions:

$V(x=0) = V(x=a) = 0$ for $y > 0$

$V(y=0) = V_0(x)$

$V \to 0$ as $y \to \infty$

Ansatz: $V = X(x)\, Y(y)$

Sep. of var. $\frac{d^2}{dx^2} X(x) = C_1 X(x)$

$\frac{d^2}{dy^2} Y(y) = C_2 Y(y)$

$C_1 + C_2 = 0$

What are C_1 and C_2 ?

$C_1 > 0$: $X(x) = A e^{\sqrt{C_1}\, x} + B^{-\sqrt{C_1}\, x}$

cannot get $X(0) = X(a) = 0$

$C_1 = 0$: $X(x) = A + Bx$

cannot get $X(0) = X(a) = 0$

$C_1 < 0$: $X(x) = A \sin kx + B \cos kx$, $k^2 = -C_1$

$C_2 = -C_1 = k^2 \Rightarrow Y(y) = C e^{ky} + D e^{-ky}$

1) $V \to 0$ as $y \to \infty$ $\Rightarrow$ $C = 0$

2) $V(x=0) = 0$ $\Rightarrow$ $B = 0$

$$\Rightarrow \quad V = \underbrace{AD}_{C} \, e^{-ky} \sin kx$$

3) $V(x=a) = 0$ $\Rightarrow$ $ka = n\pi$, $n = 1, 2, \ldots$

label solutions with n

$$\boxed{V_n(x,y) = C_n \, e^{\frac{n\pi}{a} y} \sin \frac{n\pi}{a} x}$$

4) coefficients C_n determined such that

$$\sum_{n=1}^{\infty} C_n \sin \frac{n\pi}{a} x = V_0(x)$$

use orthogonality of $\sin \frac{n\pi}{a} x$

$$\int_0^a \sin \frac{n\pi}{a} x \cdot \sin \frac{m\pi}{a} x \, dx = \begin{cases} 0 & \text{if } n \neq m \\ \frac{a}{2} & \text{if } n = m \end{cases}$$

$$\Rightarrow \quad C_n = \frac{2}{a} \int_0^a V_0(x) \sin \frac{n\pi}{a} x$$

What about other geometries?

In spherical coordinates (r, θ, φ)

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) +$$

$$+ \underbrace{\cancel{\frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2} V}}_{\text{assume } 0} = 0$$

general solution given by spherical harmonics

limit to the case when V has <u>no</u> φ-dependence

Ansatz: $V(r,\theta) = R(r)\,\Theta(\theta)$

$$\Rightarrow \frac{1}{r^2}\left(\frac{d}{dr} r^2 \frac{d}{dr} R(r)\right)\Theta(\theta) + \frac{1}{r^2} R(r) \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, \Theta(\theta)\right) = 0$$

divide with $R(r)\,\Theta(\theta)$ and mult. with r^2

$$\Rightarrow \underbrace{\frac{1}{R(r)}\frac{d}{dr}\left(r^2 \frac{d}{dr} R(r)\right)}_{= \ell(\ell+1)} + \underbrace{\frac{1}{\Theta(\theta)}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\, \Theta(\theta)\right)}_{= -\ell(\ell+1)\,,\ \ell \geq 0} = 0$$

limited for all θ

with solutions

$$R(r) = A_\ell r^\ell + B_\ell \frac{1}{r^{\ell+1}}$$

$$\Theta(\theta) = P_\ell(\cos\theta)\,,$$ Legendre polynomials

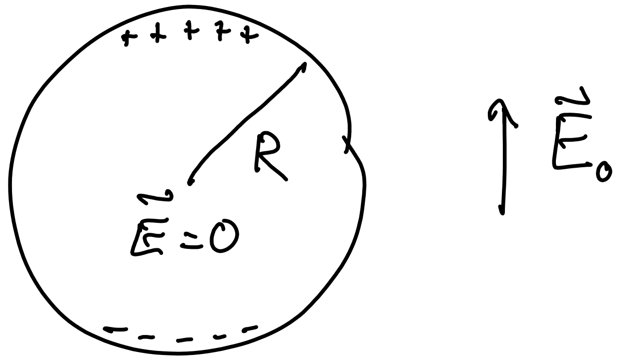
$P_0(x) = 1\,,\ P_1(x) = x\,,$

$P_2(x) = \frac{1}{2}(3x^2 - 1), \ldots$

$$\int_{-1}^{1} P_\ell(x) P_{\ell'}(x)\, dx = \frac{2}{2\ell+1}\delta_{\ell\ell'}$$

orthogonal on $-1 \leq \cos\theta \leq 1$

Example: Start with a uniform electric field $\vec{E} = E_0 \hat{z}$. Add an uncharged metal sphere (radius R) in the field. Calculate the resulting potential outside the sphere



the charges in the sphere will move until the field inside vanishes – the sphere becomes polarized

General solution (spherical symmetry)

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l \frac{1}{r^{l+1}} \right) P_l(\cos\theta)$$

proper boundary conditions?

know $V(r<R, \theta) = V_0 = \text{const}$ (equipot.)

for simplicity pick $V_0 = 0$

also know $V(r\to\infty, \theta) = -E_0 z = -E_0 r\cos\theta$

1) $r \to \infty$: $\displaystyle\sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) = -E_0 r\cos\theta$

$$A_1 = -E_0 \,, \quad A_l = 0 \quad l \neq 1$$

2, $r = R$: $A_1 R \cos\theta + \sum_{\ell} B_\ell \frac{1}{R^{\ell+1}} P_\ell(\cos\theta) = 0$

$$B_1 = -A_1 R^3 \quad , \quad B_\ell = 0 \quad \ell \neq 1$$

$$\therefore \quad V(r,\theta) = -E_0 \left(r - \frac{R^3}{r^2} \right) \cos\theta$$

external field — induced charge on sphere

electric field

$$\vec{E} = E_r \hat{r} + E_\theta \hat{\theta}$$

$$E_r = -\frac{\partial V}{\partial r}\Big|_{r=R} = E_0 \left(1 + 2\frac{R^3}{r^3}\right) \cos\theta$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta}\Big|_{r=R} = -E_0 \left(1 - \frac{R^3}{r^3}\right) \sin\theta$$

induced surface charge

$$\sigma(\theta) = \varepsilon_0 \left(E_r^{out} - E_r^{in}\right)\Big|_{r=R} =$$

$$= \varepsilon_0 E_0 (1+2) \cos\theta - 0 = 3\varepsilon_0 E_0 \cos\theta$$

also note that

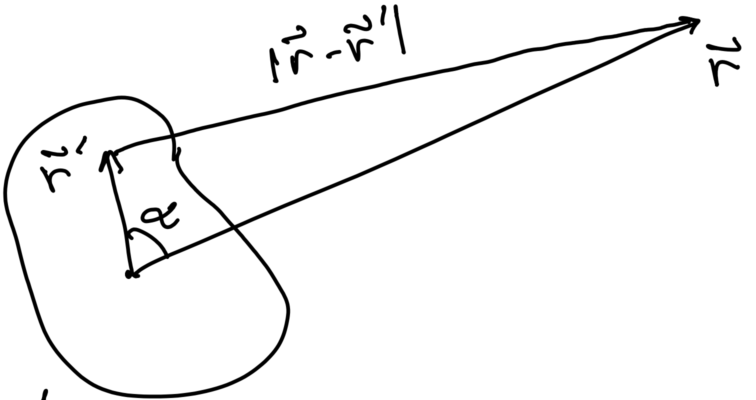
$$\left(E_\theta^{out} - E_\theta^{in}\right)\Big|_{r=R} = 0 - 0 = 0$$

Multipole expansion

method for calculating

$$V = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d^3\vec{r}\,'$$

approximately. Far away from a localized charge distribution it will look like a point charge



rewrite $\frac{1}{|\vec{r}-\vec{r}\,'|}$ as a power expansion in $\frac{1}{|\vec{r}|}$

$$(\vec{r}-\vec{r}\,')^2 = \vec{r}^{\,2} + \vec{r}\,'^2 - 2\vec{r}\cdot\vec{r}\,' = r^2 + r'^2 - 2rr'\cos\alpha$$

$$\Rightarrow |\vec{r}-\vec{r}\,'| = r\Big(1 + \underbrace{\frac{r'^2}{r^2} - 2\frac{r'}{r}\cos\alpha}_{\delta \ll 1}\Big)^{1/2} = r\left(1+\frac{\delta}{2}+\dots\right)$$

$$\Rightarrow \frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{r}\left(1 - \frac{\delta}{2} + \frac{3}{8}\delta^2 - \frac{5}{16}\delta^3 + \dots\right)$$

$$= \frac{1}{r}\left[1 - \frac{1}{2}\frac{r'}{r}\left(\frac{r'}{r} - 2\cos\alpha\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos\alpha\right)^2 + \dots\right]$$

$$= \frac{1}{r}\Big[1 - \frac{r'}{r}\underbrace{\cos\alpha}_{P_1(\cos\alpha)} + \left(\frac{r'}{r}\right)^2 \underbrace{\frac{3\cos^2\alpha - 1}{2}}_{P_2(\cos\alpha)} + \dots$$

In fact

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)$$

$$\because \quad V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int (r')^n P_n(\cos\alpha)\, \rho(\vec{r}\,')\, d^3\vec{r}\,' =$$

$$= \frac{1}{4\pi\varepsilon_0}\Bigg[\frac{1}{r} \underbrace{\int \rho(\vec{r}\,')\, d^3\vec{r}\,'}_{Q\text{ - monopole}} + \frac{1}{r^2} \underbrace{\int r'\cos\alpha\, \rho(\vec{r}\,')\, d^3\vec{r}\,'}_{\hat{r}\cdot \underbrace{\int \vec{r}\,' \rho(\vec{r}\,')\, d^3\vec{r}\,'}_{\vec{p}\text{ - dipole}}} + \underset{\substack{\uparrow \\ \text{quadro-} \\ \text{pole}}}{\ldots} \Bigg]$$

$$= \underbrace{\frac{Q}{4\pi\varepsilon_0 r}}_{V_{mon}} + \underbrace{\frac{\vec{p}\cdot\hat{r}}{4\pi\varepsilon_0 r^2}}_{V_{dip}} + \underbrace{\frac{\sum Q_{ij}\, \hat{r}_i \hat{r}_j}{4\pi\varepsilon_0 r^3}}_{V_{quad}} + \ldots$$

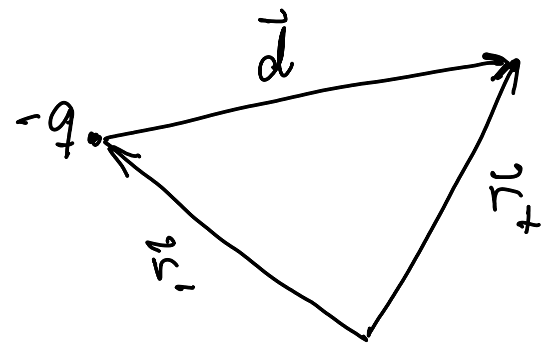
To be more concrete:

- consider a physical dipole with

$$\rho_{dipole} = q\,\delta(\vec r' - \vec r_+) - q\,\delta(\vec r' - \vec r_-)$$

$$\Rightarrow \quad Q = \int \rho_{dipole}(\vec r')\, d^3\vec r' = q - q = 0$$

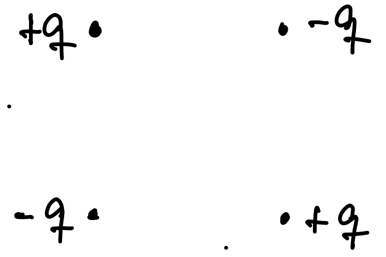
$$\vec p = \int \vec r'\, \rho_{dipole}(\vec r')\, d^3\vec r' = q\underbrace{(\vec r_+ - \vec r_-)}_{\vec d}$$



perfect dipole:
$d \to 0$, $q \to \infty$
$p = qd$ fixed

$Q_{ij} \approx 0$

- and a physical quadrupole



or $\quad \vec p \Uparrow \qquad \Downarrow -\vec p$

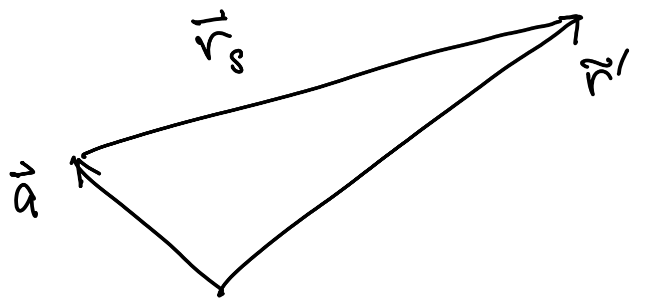
$$\Rightarrow \quad Q = 0, \qquad \vec p = 0$$

Choice of origin?

- Q indep of origin (follows from def)
- if $Q = 0$ then $\vec p$ is indep. of origin
- etc

Ex. $\vec{p}_s = \int \vec{r}_s \, \rho(\vec{r}\,') \, d^3\vec{r}\,' = [\vec{r}\,' = \vec{r}_s + \vec{a}] =$

$$= \underbrace{\int \vec{r}\,' \rho(\vec{r}\,') \, d^3\vec{r}\,'}_{\vec{p}} - \vec{a} \underbrace{\int \rho(\vec{r}\,') \, d^3\vec{r}\,'}_{Q}$$



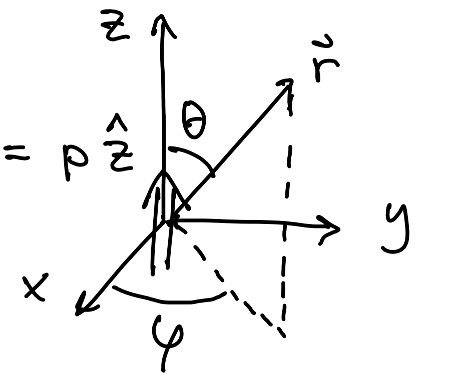
Electric field

$$\vec{E} = -\vec{\nabla} V = -\vec{\nabla} V_{mon} - \vec{\nabla} V_{dip} - \vec{\nabla} V_{quad} - \dots$$

- we already know $\vec{E}_{mon} = -\vec{\nabla} V_{mon}$

what about $\vec{E}_{dipole}$?

$\vec{p} = p\hat{z}$



$$\vec{E}_{dipole} = -\vec{\nabla} \frac{\hat{r} \cdot \vec{p}}{4\pi\varepsilon_0 r^2}$$

$$= -\vec{\nabla} \frac{p \cos\theta}{4\pi\varepsilon_0 r^2}$$

$$= \frac{2p\cos\theta}{4\pi\varepsilon_0 r^3} \hat{r} + \frac{p \sin\theta}{4\pi\varepsilon_0 r^3} \hat{\theta} + 0 \hat{\varphi}$$

[prob 3.36] $= \dfrac{3(\vec{p} \cdot \hat{r})\hat{r} - \vec{p}}{4\pi\varepsilon_0 r^3}$ coordinate free form

and $\vec{E}_{quad}$?

$$\vec{E}_{quad} \sim \frac{1}{r^4}$$