Potential formulation:

Maxwell's eqn's:

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\rho \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}\vec{B}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial}{\partial t}\vec{E}$$

we have already seen that

$$\vec{E} = -\vec{\nabla}V - \frac{\partial}{\partial t}\vec{A} \qquad \left(\text{follows from } \vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}\vec{B}\right)$$

$$\vec{B} = \vec{\nabla}\times\vec{A} \qquad \left(\text{follows from } \vec{\nabla}\cdot\vec{B} = 0\right)$$

Gauss's law then becomes

$$-\vec{\nabla}^2 V - \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A} = \frac{1}{\varepsilon_0}\rho$$

this can be written in a manifestly Lorentz-invariant way by introducing

$$X = \underbrace{\varepsilon_0\mu_0\frac{\partial}{\partial t}}_{\frac{1}{c^2}\frac{\partial}{\partial t} = \frac{1}{c}\frac{\partial}{\partial x^0}} V + \vec{\nabla}\cdot\vec{A} = \partial_\mu\left(\frac{1}{c}V, \vec{A}\right) = \partial_\mu A^\mu$$

$$\partial^\mu = \left(\frac{\partial}{\partial x^0}, -\vec{\nabla}\right)$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{A}\right) = \frac{\partial}{\partial t}\partial_\nu A^\nu - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}V$$

which in turn gives Gauss's law as

$$\underbrace{\left(\frac{1}{c^2}\frac{\partial^2}{\partial t} - \vec{\nabla}^2\right)}_{\partial^2} V - \frac{\partial}{\partial t}\partial_\nu A^\nu = \frac{1}{\varepsilon_0}\rho$$

Next we consider Ampere's law

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial}{\partial t}\vec{E}$$

$$\underbrace{\vec{\nabla}\times(\vec{\nabla}\times\vec{A})}_{\vec{\nabla}(\vec{\nabla}\cdot\vec{A})-\vec{\nabla}^2\vec{A}} = \mu_0\vec{J} + \mu_0\varepsilon_0\underbrace{\frac{\partial}{\partial t}\left(-\vec{\nabla}V-\frac{\partial}{\partial t}\vec{A}\right)}_{-\mu_0\varepsilon_0\vec{\nabla}\frac{\partial}{\partial t}V-\mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\vec{A}}$$

$$\Rightarrow \underbrace{\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\vec{\nabla}^2\right)}_{\partial^2}\vec{A} + \vec{\nabla}\underbrace{\left(\frac{1}{c^2}\frac{\partial}{\partial t}V+\vec{\nabla}\cdot\vec{A}\right)}_{\partial_\nu A^\nu} = \mu_0\vec{J}$$

$$\therefore\ \partial^2\vec{A} + \vec{\nabla}\partial_\nu A^\nu = \mu_0\vec{J}$$

Introducing a four-current $J^\mu = (c\rho, \vec{J})$ we can combine Gauss's and Ampere's laws

$$\partial^2\underbrace{\left(\frac{1}{c}V,\vec{A}\right)}_{A^\mu} - \underbrace{\left(\frac{1}{c}\frac{\partial}{\partial t},-\vec{\nabla}\right)}_{\partial^\mu}\partial_\nu A^\nu = \mu_0\underbrace{(c\rho,\vec{J})}_{J^\mu}$$

$$\therefore\ \partial^2 A^\mu - \partial^\mu\partial_\nu A^\nu = \mu_0 J^\mu$$

this can also be written as

$$\partial_\nu\underbrace{\left(\partial^\nu A^\mu - \partial^\mu A^\nu\right)}_{F^{\nu\mu}} = \mu_0 J^\mu$$

so that

$$\boxed{\partial_\nu F^{\nu\mu} = \mu_0 J^\mu}$$

Maxwell's eqn's

note:

$$F^{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{c}E_x & \frac{1}{c}E_y & \frac{1}{c}E_z \\ -\frac{1}{c}E_x & 0 & B_z & -B_y \\ -\frac{1}{c}E_y & -B_z & 0 & B_x \\ -\frac{1}{c}E_z & B_y & -B_x & 0 \end{pmatrix}$$

immediately see that

$$\underbrace{\partial_\mu \partial_\nu}_{\text{symm}} \underbrace{F^{\nu\mu}}_{\text{anti-symm}} = \mu_0 \partial_\mu J^\mu = 0 \quad \Rightarrow \quad \frac{\partial}{\partial t} \rho + \vec{\nabla} \cdot \vec{J} = 0$$

continuity eqn !

Noether's theorem:

$$\partial_\mu J^\mu = 0$$

there is a conserved charge (the electric one)

<u>Lorentz transformations:</u>

The four-vector potential formulation is Lorentz covariant

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

$$F^{\mu\nu} \to F'^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma F^{\rho\sigma}$$

$$A^\mu \to A'^\mu = \Lambda^\mu{}_\nu A^\nu$$

$$J^\mu \to J'^\mu = \Lambda^\mu{}_\nu J^\nu$$

Gauge-transformations

If the four-vector potential A^μ is changed according to

$$A^\mu = \left(\frac{1}{c}V, \vec{A}\right) \rightarrow A'^\mu = \left(\frac{1}{c}V - \frac{1}{c}\frac{\partial}{\partial t}\Lambda, \vec{A} + \vec{\nabla}\Lambda\right)$$

where Λ is a arbitrary fcn of $\vec{r}$ and t then the $\vec{E}$ and $\vec{B}$-fields are unchanged

$$\vec{E} \rightarrow \vec{E}' = -\vec{\nabla}\left(V - \frac{\partial}{\partial t}\Lambda\right) - \frac{\partial}{\partial t}\left(\vec{A} + \vec{\nabla}\Lambda\right) =$$

$$= -\vec{\nabla}V - \frac{\partial}{\partial t}\vec{A} = \vec{E}$$

$$\vec{B} \rightarrow \vec{B}' = \vec{\nabla} \times \left(\vec{A} + \vec{\nabla}\Lambda\right) \underset{\uparrow}{=} \vec{\nabla} \times \vec{A} = \vec{B}$$

$$\left[\vec{\nabla} \times (\vec{\nabla}\Lambda) = 0\right]$$

On four-vector form

$$A^\mu \rightarrow A'^\mu = A^\mu - \partial^\mu \Lambda$$

$$F^{\mu\nu} \rightarrow F'^{\mu\nu} = \partial^\mu(A^\nu - \partial^\nu\Lambda) - \partial^\nu(A^\mu - \partial^\mu\Lambda)$$

$$= \partial^\mu A^\nu - \partial^\nu A^\mu = F^{\mu\nu}$$

where as

$$\partial_\mu A^\mu \rightarrow \partial_\mu A'^\mu = \partial_\mu A^\mu - \partial^2\Lambda$$

$\therefore$ free to choose Λ such that (for example)

$$\partial_\mu A'^\mu = 0 \qquad \text{Lorenz gauge}$$

$$\vec{\nabla} \cdot \vec{A}' = 0 \qquad \text{Coulomb gauge}$$

Coulomb gauge:

The gauge condition

$$\vec{\nabla}\cdot\vec{A}' = 0 \quad \Rightarrow \quad \partial_\mu A'^\mu = \frac{1}{c^2}\frac{\partial}{\partial t}V'$$

together with

$$\partial^2 A'^\mu - \partial^\mu \partial_\nu A'^\nu = \mu_0 (c\rho, \vec{J})$$

gives for the zero-component ($\mu=0$):

$$\partial^2 \frac{1}{c}V' - \frac{\partial}{\partial x_0}\frac{1}{c^2}\frac{\partial}{\partial t}V' = \mu_0 c\rho = \frac{1}{c\varepsilon_0}\rho$$

$$\Rightarrow \quad \vec{\nabla}^2 V' = -\frac{1}{\varepsilon_0}\rho \qquad \text{same as in electrostatics!}$$

and for the three-vector component ($\mu=i$):

$$\partial^2 \vec{A}' + \frac{1}{c^2}\vec{\nabla}\frac{\partial}{\partial t}V = \mu_0 \vec{J} \qquad \text{ugly!}$$

Lorenz gauge:

From $\partial_\nu A'^\nu = 0$ it follows that

$$\partial^2 A'^\mu = \mu_0 J^\mu$$

– the inhomogeneous wave equation

We have already studied the plane-wave solutions for $\rho, \vec{J} = 0$ (drop ′)

$$\tilde{A}^\mu(\vec{r},t) = \tilde{A}^\mu_\pm \, e^{i(\vec{k}\cdot\vec{r} \pm \omega t)}$$

In addition we have spherical wave solu's

$$\tilde{A}^\mu(\vec{r},t) = \tilde{A}^\mu_\pm \, \frac{e^{i(kr \pm \omega t)}}{r}$$

moving $\left\{ \begin{matrix} \text{into } (+) \\ \text{out from } (-) \end{matrix} \right\}$ the origin

⌈ Laplace in spherical coordinates (r,θ,φ) assuming spherical symmetry:

$$\vec{\nabla}^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} f\right) = \frac{1}{r^2} 2r \frac{\partial}{\partial r} f + \frac{\partial^2}{\partial r^2} f =$$

$$= \frac{1}{r}\left(2\frac{\partial}{\partial r} f + r\frac{\partial^2}{\partial r^2} f\right) = \frac{1}{r}\left(\frac{\partial}{\partial r} f + r\frac{\partial^2}{\partial r^2} f + \frac{\partial}{\partial r} f\right)$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r} f + f\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}(rf)\right)$$

$$\partial^2 \tilde{A}^\mu = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right)\tilde{A}^\mu = 0$$

$$\Rightarrow \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\tilde{A}^\mu) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\tilde{A}^\mu$$

$$\therefore \frac{\partial^2}{\partial r^2}(r\tilde{A}^\mu) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(r\tilde{A}^\mu)$$ ⌋

Retarded potential:

To solve

$$\partial^2 A^\mu = \mu_0 J^\mu$$

generally we use the method of Green's fcn's, suppose we have a fcn G that fulfills

$$\underbrace{\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right)}_{\partial_r^2} G(\vec{r},t;\vec{r}',t') = 4\pi \underbrace{\delta^{(3)}(\vec{r}-\vec{r}')\,\delta(t-t')}_{\delta^{(4)}(r-r')}$$

$$\partial_r^2\, G(r,r')$$

then the solution is

$$A^\mu(\vec{r},t) = \frac{\mu_0}{4\pi}\int J^\mu(\vec{r}',t')\,G(\vec{r},t;\vec{r}',t')\,d^4r$$

there are two possible solutions for G

$$G_\pm(r,r') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{\pm i|\vec{k}||\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}\,e^{-i\omega(t-t')}\,d\omega$$

assuming that we have no dispersion, $\omega = |\vec{k}|c$

$$\Rightarrow G_\pm(r,r') = \frac{1}{|\vec{r}-\vec{r}'|}\underbrace{\frac{1}{2\pi}\int e^{i\omega\left(t'-t\pm\frac{|\vec{r}-\vec{r}'|}{c}\right)}\,d\omega}_{\delta\left[t'-\left(t\mp\frac{|\vec{r}-\vec{r}'|}{c}\right)\right]}$$

and we get two solutions for A^μ

$$A^\mu_\pm(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{J^\mu\left(\vec{r}',\, t\mp\frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|}\,d^3\vec{r}'$$

$\therefore$ the EM fields at $(\vec{r}, t)$ depend on the source at the time $t \mp \frac{|\vec{r}-\vec{r}'|}{c}$ where the $\left\{\begin{matrix}-\\+\end{matrix}\right\}$ sign corresponds to the $\left\{\begin{matrix}\text{retarded}\\ \text{advanced}\end{matrix}\right\}$ solutions

causality tells us that the retarded sol'n is the physical one

$$\therefore \quad \boxed{A^{\mu}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \frac{J^{\mu}\left(\vec{r}', t - \frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|} d^3\vec{r}'}$$

(can also add solutions to hom. equ. $\partial^2 A^{\mu} = 0$)

To solve this we will make some simplifying assumptions as we go along

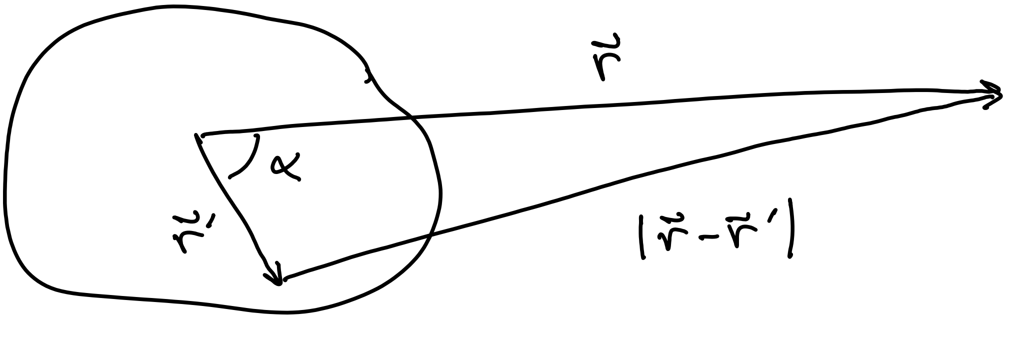
1) Assume that the source J^{μ} is localised and has harmonic time-dependence

$$\Rightarrow \tilde{A}^{\mu}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\tilde{J}^{\mu}(\vec{r}')}{|\vec{r}-\vec{r}'|} e^{-i\omega\left(t - \frac{|\vec{r}-\vec{r}'|}{c}\right)} d^3\vec{r}'$$

2) to calculate the integral we make a Taylor expansion (cf multipole exp.)

$$|\vec{r}-\vec{r}'| = |\vec{r}| - \frac{\vec{r}\cdot\vec{r}'}{|\vec{r}|} + \ldots = r - r'\cos\alpha + \ldots$$

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{|\vec{r}|} + \frac{\vec{r}\cdot\vec{r}'}{|\vec{r}|^3} + \ldots = \frac{1}{r} + \frac{r'\cos\alpha}{r^2} + \ldots$$



and use the continuity equation for the source

$$\frac{\partial}{\partial t'}\left(\tilde{\rho}(\vec{r}')\,e^{-i\omega t'}\right) = -\vec{\nabla}'\cdot\tilde{\vec{J}}(\vec{r}')\,e^{-i\omega t'}$$

$$\Rightarrow \quad i\omega\,\tilde{\rho}(\vec{r}') = \vec{\nabla}'\cdot\tilde{\vec{J}}(\vec{r}')$$

Leading terms in $\tilde{A}^\mu$: (monopole)

$\underline{\mu=0}$: $$\tilde{A}^0_m(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{c\tilde{\rho}(\vec{r}')}{r}\,e^{-i\omega(t-\frac{r}{c})}\,d^3\vec{r}'$$

$$\Rightarrow \tilde{V}_m(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\,\frac{e^{i(kr-\omega t)}}{r}\;\underbrace{\int\tilde{\rho}(\vec{r}')\,d^3\vec{r}'}\;=0$$

$$[\omega\neq 0]\quad \int\frac{1}{i\omega}\,\vec{\nabla}'\cdot\tilde{\vec{J}}(\vec{r}')\,d^3\vec{r}'$$

$$= \frac{1}{i\omega}\oint\tilde{\vec{J}}^\mu(\vec{r}')\cdot d\vec{S}' = 0$$

since $\tilde{\vec{J}}$ is localized

(electric dipole)

$$\tilde{\vec{A}}_{ed}(\vec{r},t) = \frac{\mu_0}{4\pi}\,\frac{e^{i(kr-\omega t)}}{r}\int\tilde{\vec{J}}(\vec{r}')\,d^3\vec{r}'$$

Now consider (Problem 5.7)

$$\vec{\nabla}'\cdot(x'\tilde{\vec{J}}) = x'\,\vec{\nabla}'\cdot\tilde{\vec{J}} + \tilde{\vec{J}}\cdot\underbrace{\vec{\nabla}'x'}_{\hat{x}'} = x'\,\vec{\nabla}'\cdot\tilde{\vec{J}} + \tilde{J}_{x'}$$

$$\Rightarrow \vec{\nabla}'\cdot\underbrace{(x'\tilde{\vec{J}}\hat{x}' + y'\tilde{\vec{J}}\hat{y}' + z'\tilde{\vec{J}}\hat{z}')}_{\vec{r}'\tilde{\vec{J}}} = \vec{r}'\,\vec{\nabla}'\cdot\tilde{\vec{J}} + \tilde{\vec{J}}$$

$$\Rightarrow \int\tilde{\vec{J}}(\vec{r}')\,d^3\vec{r}' = \underbrace{\int\vec{\nabla}'\cdot(\vec{r}'\tilde{\vec{J}})\,d^3\vec{r}'}_{\int\vec{r}'(\tilde{\vec{J}}\cdot d\vec{S})\;=\;0} - \underbrace{\int\vec{r}'\,\vec{\nabla}'\cdot\tilde{\vec{J}}\,d^3\vec{r}}_{-i\omega\underbrace{\int\vec{r}'\tilde{\rho}(\vec{r}')\,d^3\vec{r}'}_{\tilde{\vec{p}}_0}}$$

$[\tilde{\vec{J}}=0$ on surface$]$ $\quad\left[\frac{\partial}{\partial t}\tilde{\rho} = -i\omega\tilde{\rho}\right]$

$$\therefore \quad \tilde{\vec{A}}_{ed} = -i\omega \frac{\mu_0}{4\pi} \frac{e^{i(kr-\omega t)}}{r} \tilde{\vec{p}}_0 \quad \propto \frac{\partial}{\partial t} \vec{p}$$

Higher order terms will give corrections (magnetic dipole, electric quadropole etc)

Since we have harmonic time-dependence we only need $\tilde{\vec{A}}$ to find $\tilde{\vec{B}}$ and $\tilde{\vec{E}}$ outside the source

$$\tilde{\vec{B}} = \vec{\nabla} \times \tilde{\vec{A}}$$

$$\tilde{\vec{E}} = \frac{i}{\omega \varepsilon_0 \mu_0} \vec{\nabla} \times \tilde{\vec{B}} \qquad \Big(\vec{\nabla} \times \tilde{\vec{B}} = \underbrace{\mu_0 \tilde{\vec{J}}}_{=0 \text{ outside}} + \mu_0 \varepsilon_0 \underbrace{\frac{\partial}{\partial t} \tilde{\vec{E}}}_{-i\omega \tilde{\vec{E}}} \Big)$$

Inserting $\tilde{\vec{A}}_{ed}$ gives

$$\left[\frac{\partial}{\partial x}\left(\frac{1}{r} e^{ikr}\right) = \frac{1}{r} ik \frac{\partial r}{\partial x} e^{ikr} - \frac{1}{r^2} \frac{\partial r}{\partial x} e^{ikr} = \right.$$

$$\left. \left[\frac{\partial r}{\partial x} = \frac{x}{r}\right] \quad = ik \frac{1}{r} e^{ikr} \left(1 + \frac{i}{kr}\right) \frac{x}{r} \right]$$

$$\tilde{\vec{B}} = \underbrace{-\frac{i\omega\mu_0}{4\pi} ik}_{\frac{\mu_0 c k^2}{4\pi} = \frac{k^2}{4\pi\varepsilon_0 c}} \frac{e^{i(kr-\omega t)}}{r} \left(1 + \frac{i}{kr}\right) \hat{r} \times \tilde{\vec{p}}_0$$

$$\tilde{\vec{E}} = (\text{half page}) = \overbrace{i \frac{1}{\omega \varepsilon_0 \mu_0} \frac{\mu_0 \omega k}{4\pi} ik}^{-\frac{k^2}{4\pi\varepsilon_0}} \frac{e^{i(kr-\omega t)}}{r}$$

$$\left[\hat{r} \times (\hat{r} \times \tilde{\vec{p}}_0) - (3\hat{r}(\hat{r} \cdot \tilde{\vec{p}}_0) - \tilde{\vec{p}}_0)\left(\frac{1}{k^2 r^2} - \frac{i}{kr}\right) \right]$$

Specialize to:

1) cloose to the source $kr \ll 1$

$$\tilde{\vec{B}} = i \frac{k}{4\pi\varepsilon_0 c} \frac{1}{r^2} \hat{r} \times \tilde{\vec{p}}_0 \, e^{-i\omega t}$$

$$\tilde{\vec{E}} = \underbrace{\frac{1}{4\pi\varepsilon_0} \left(3\hat{r}(\hat{r}\cdot\tilde{\vec{p}}_0) - \tilde{\vec{p}}_0\right) \frac{1}{r^3}}_{\text{field from static electric dipole}} e^{-i\omega t}$$

$$|\vec{B}| \sim \frac{kr}{c} |\vec{E}| \ll |\vec{E}|$$

∴ $\vec{E}$ - field dominates

($|\vec{B}| \to 0$ as $kc = \omega \to 0$, static limit)

2) far from the source, $kr \gg 1$
(radiation zone)

$$\tilde{\vec{B}} = \frac{k^2}{4\pi\varepsilon_0 c} \frac{e^{i(kr-\omega t)}}{r} \underbrace{\hat{r} \times \tilde{\vec{p}}_0}_{-\sin\theta\,\hat{\varphi}\,|\tilde{\vec{p}}_0|} \quad \text{if } \tilde{\vec{p}}_0 = |\tilde{\vec{p}}_0|\hat{z}$$

$$\tilde{\vec{E}} = \frac{k^2}{4\pi\varepsilon_0} \frac{e^{i(kr-\omega t)}}{r} \underbrace{(\hat{r} \times \tilde{\vec{p}}_0) \times \hat{r}}_{-\sin\theta\,\hat{\theta}\,|\tilde{\vec{p}}_0|} \quad \text{—— " ——}$$

radiated intensity:

$$\langle \vec{S} \rangle = \frac{1}{2\mu_0} \tilde{\vec{E}} \times \tilde{\vec{B}}^* = \frac{ck^4}{32\pi^2\varepsilon_0} \frac{1}{r^2} \underbrace{\left[(\hat{r} \times \tilde{\vec{p}}_0) \times \hat{r}\right] \times (\hat{r} \times \tilde{\vec{p}}_0^*)}_{(\hat{r} \times \vec{p}_0)^2\,\hat{r}}$$

$$= \frac{\mu_0 \omega^4}{32\pi^2 c r^2} \underbrace{(\hat{r} \times \vec{p}_0)^2}_{|\vec{p}_0|^2 \sin^2\theta} \hat{r} \quad , \quad \vec{p}_0 = |\vec{p}_0|\,\hat{z}$$

and the radiated power is

$$\langle P \rangle = \int \langle \vec{S} \rangle \cdot \hat{r}\, r^2 d\Omega =$$

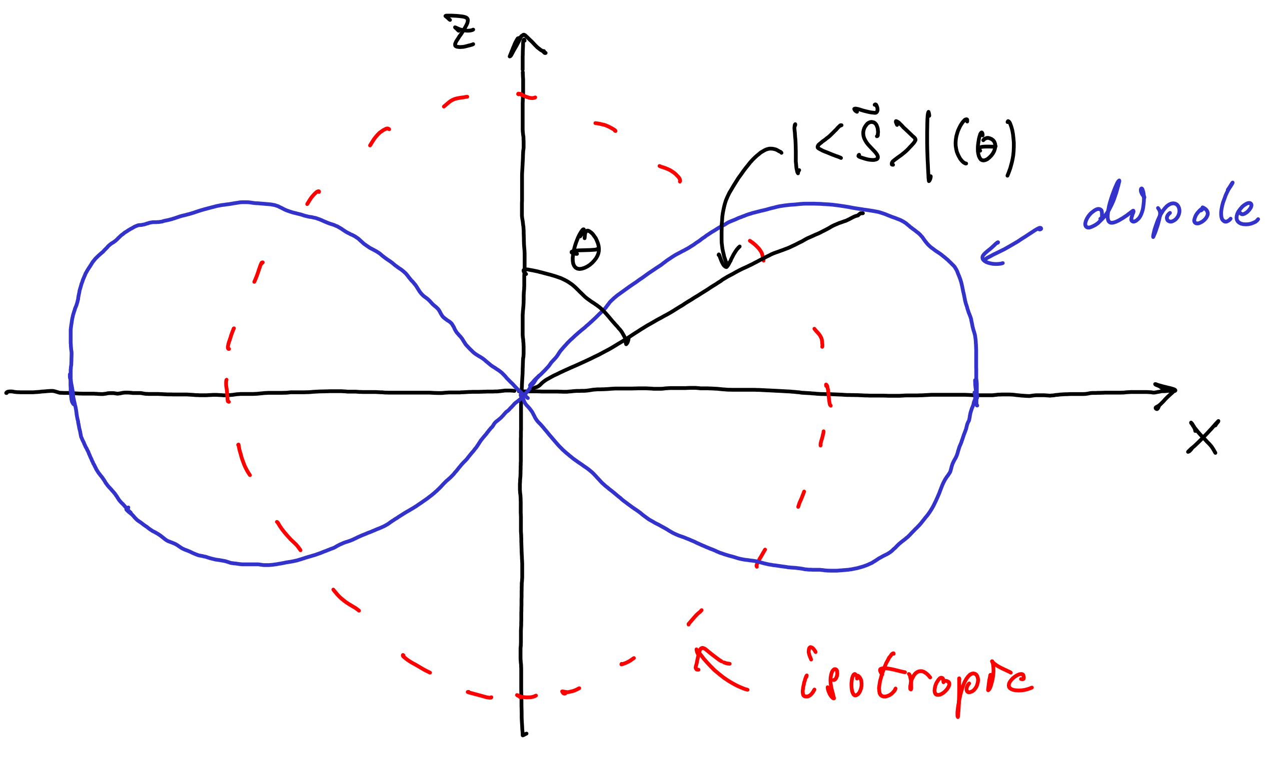
$$= \frac{\mu_0 \omega^4}{32\pi^2 c} |\vec{p}_0|^2 \underbrace{\int \sin^2\theta \, d(\cos\theta)\, d\varphi}_{2\pi \left[\cos\theta - \frac{\cos^3\theta}{3}\right]_{-1}^{1} = \frac{8\pi}{3}}$$

$$= \frac{\mu_0 \omega^4}{12\pi c} |\vec{p}_0|^2 = \underbrace{\sqrt{\frac{\mu_0}{\varepsilon_0}}}_{Z_0} \frac{\omega^4}{12\pi c^2} |\vec{p}_0|^2$$

note: $|\langle \vec{S} \rangle| = \langle P \rangle \cdot \underbrace{\frac{3}{8\pi r^2} \sin^2\theta}_{\text{would be } = \frac{1}{4\pi r^2} \text{ if isotropic}}$

$$|\langle \vec{S} \rangle_{max}| = \frac{3}{2} \frac{1}{4\pi r^2} \langle P \rangle$$

Antenna diagram describes ang. dist.



[More generally we can also write

$$\vec{p}_0 e^{i(kr-\omega t)} = \vec{p}_0 e^{-i\omega[t-\frac{r}{c}]} = \vec{p}(t')$$

so that

$$\omega^2 \vec{p}_0 e^{i(kr-\omega t)} = -\frac{\partial^2}{\partial t^2}\vec{p}(t') = -\ddot{\vec{p}}(t')$$

and we get $(\vec{p}_0 = |\vec{p}_0|\,\hat{z})$ non-relativistic!

$$\vec{B}(\vec{r},t) = \frac{\mu_0}{4\pi c}\frac{1}{r}\ddot{\vec{p}}(t')\sin\theta\,\hat{\varphi}$$

$$\vec{E}(\vec{r},t) = \frac{\mu_0}{4\pi}\frac{1}{r}\ddot{\vec{p}}(t')\sin\theta\,\hat{\theta}$$

$$P(t) = \frac{\mu_0}{6\pi c}|\ddot{\vec{p}}(t')|^2 \quad \text{(instantaneous!)}$$

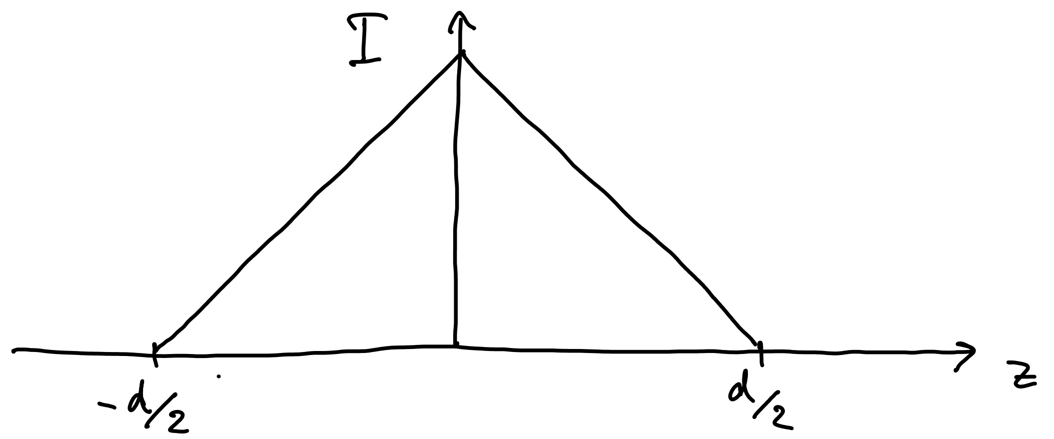
Example: oscillating point charge

$$\vec{p} = q\vec{r} \Rightarrow \ddot{\vec{p}} = q\ddot{\vec{r}} \Rightarrow P = \frac{\mu_0 q^2}{6\pi c}|\ddot{\vec{r}}(t')|^2$$

∴ accelerated charges radiate]

Simple model of an antenna:

$$\tilde{\vec{J}}^{\mu}(\vec{r}\,') = I_0 \left(1 - \frac{2|z'|}{d}\right) \delta(x')\,\delta(y')\,\hat{z}\, e^{-i\omega t}$$



need to calculate $\tilde{\vec{p}} = \tilde{\vec{p}}_0 e^{-i\omega t}$

continuity equ: $\vec{\nabla}\cdot\tilde{\vec{J}} = -\frac{\partial}{\partial t}\tilde{\rho} = i\omega\tilde{\rho}$

$$\Rightarrow \tilde{\rho} = \frac{z'}{|z'|}\frac{2 i I_0}{\omega d}\,\delta(x')\,\delta(y')\,e^{-i\omega t} = \tilde{\rho}_0 e^{-i\omega t}$$

the electric dipole moment

$$\tilde{\vec{p}}_0 = \int_{-d/2}^{d/2} \vec{r}\,'\,\tilde{\rho}_0(\vec{r}\,')\,d^3\vec{r}\,' = 2\int_0^{d/2} z'\,\frac{2 i I_0}{\omega d}\,dz'\,\hat{z}$$

$$= \frac{4 i I_0}{\omega d}\left[\frac{z'^2}{2}\right]_0^{d/2}\hat{z} = \frac{i I_0 d}{2\omega}\,\hat{z}$$

the radiated power is then given by

$$\langle P\rangle = \frac{\mu_0\omega^4}{12\pi c}|\tilde{p}_0|^2 = \frac{\mu_0\omega^4}{12\pi c}\left(\frac{I_0 d}{2\omega}\right)^2 = \frac{\mu_0 c k^2 d^2 I_0^2}{48\pi}$$

$[\omega = kc]$

compare with $\langle P\rangle = \frac{1}{2} R I_0^2$

$\Rightarrow$ antenna resistance $R = \frac{\mu_0 c k^2 d^2}{24\pi} \approx 5.0\,(kd)^2\,\text{ohm}$

$[\mu_0 c = Z_0 = 377\,\text{ohm}]$