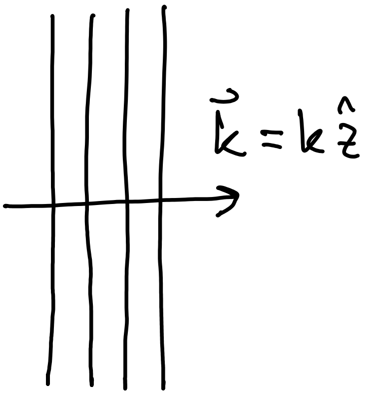
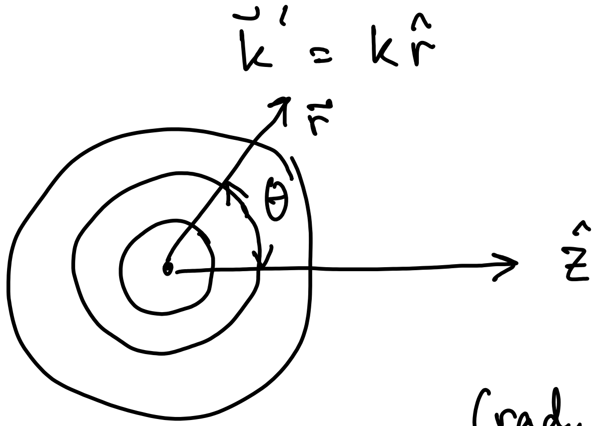
Scattering: (elastic)





(rad. zone)

incoming plane wave

$$\tilde{\vec{E}}_p = \hat{e}_i \tilde{E}_0 e^{i(kz-\omega t)}$$

$$\tilde{\vec{B}}_p = \frac{1}{c}\hat{k} \times \tilde{\vec{E}}_p$$

note: $kz = kr\cos\theta$

outgoing spherical wave

$$\tilde{\vec{E}}_s = E_0 \tilde{\vec{f}}(\theta,\varphi) \frac{e^{i(kr-\omega t)}}{kr}$$

$\uparrow$ scattering amplitude

$$\tilde{\vec{B}}_s = \frac{1}{c}\hat{r} \times \tilde{\vec{E}}_s$$

adding up we get:

$$\tilde{\vec{E}} = \tilde{E}_0 \left[\hat{e}_i e^{ikr\cos\theta} + \tilde{\vec{f}}(\theta,\varphi) \frac{e^{ikr}}{kr} \right] e^{-i\omega t}$$

$$\tilde{\vec{B}} = \frac{1}{c}\tilde{E}_0 \left[\hat{z} \times \hat{e}_i e^{ikr\cos\theta} + (\hat{r} \times \tilde{\vec{f}}(\theta,\varphi)) \frac{e^{ikr}}{kr} \right] e^{-i\omega t}$$

and

$$\langle \tilde{\vec{S}} \rangle = \frac{1}{4\mu_0} (\tilde{\vec{E}} \times \tilde{\vec{B}}^* + \tilde{\vec{E}}^* \times \tilde{\vec{B}})$$

so that

$$\int \langle \tilde{\vec{S}} \rangle \cdot \hat{r}\, r^2 d\Omega = \frac{1}{4\mu_0 c} |\tilde{E}_0|^2 \int \Big\{ \Big(\hat{e}_i e^{ikr\cos\theta} + \tilde{\vec{f}} \frac{e^{ikr}}{kr} \Big)$$
$$\times \Big(\hat{z} \times \hat{e}_i e^{-ikr\cos\theta} + \hat{r} \times \tilde{\vec{f}}^* \frac{e^{-ikr}}{kr} \Big) + \text{h.c.} \Big\} \cdot \hat{r}\, r^2 d\Omega$$

'incoming wave

$kr \gg 1 \Rightarrow$ average to 0 except for $\theta = 0$

$$= \frac{1}{4\mu_0 c}|\tilde{E}_0|^2 \int \Big\{ \cancel{\hat{z}} + \vec{f} \times (\hat{z}\times\hat{e}_i) \cancel{\frac{e^{ikr(1-\cos\theta)}}{kr}}^{\to 0} +$$

$$\hat{e}_i \times (\hat{r}\times\vec{f}^*) \cancel{\frac{e^{-ikr(1-\cos\theta)}}{kr}}^{\to 0} + \frac{\vec{f}\times(\hat{r}\times\vec{f}^*)}{k^2 r^2} + h.c. \Big\} \cdot \hat{r}\, r^2 d\Omega$$

$$= \underbrace{\frac{1}{2\mu_0 c}|\tilde{E}_0|^2}_{\langle \tilde{\vec{S}}_p \rangle} \int \underbrace{\frac{1}{k^2}\left(|\vec{f}(\theta,\varphi)|^2 - |\vec{f}(\theta,\varphi)\cdot\hat{r}|^2 \right)}_{\equiv \frac{d\sigma}{d\Omega}} d\Omega$$

differential cross-section !

Example: scattering from electric dipole

identifying $\tilde{E}_0 \frac{\vec{f}(\theta,\varphi)}{kr} = \frac{k^2}{4\pi\varepsilon_0 r} \underbrace{(\hat{r}\times\tilde{\vec{p}}_0)\times\hat{r}}_{\tilde{\vec{p}}_0 - \hat{r}(\tilde{\vec{p}}_0\cdot\hat{r})}$

$$\Rightarrow \vec{f}(\theta,\varphi) = \frac{1}{E_0}\frac{k^3}{4\pi\varepsilon_0}(\hat{r}\times\tilde{\vec{p}}_0)\times\hat{r},$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{|\tilde{E}_0|^2}\frac{k^4}{16\pi^2\varepsilon_0^2}\Big[\underbrace{|(\hat{r}\times\tilde{\vec{p}}_0)\times\hat{r}|^2}_{} - |\underbrace{[(\hat{r}\times\tilde{\vec{p}}_0)\times\hat{r}]\cdot\hat{r}}_{=0}|^2 \Big]$$

$$|\tilde{\vec{p}}_0|^2 - |\tilde{\vec{p}}_0\cdot\hat{r}|^2 = [(\hat{r}\times\tilde{\vec{p}}_0)\times\hat{r}]\cdot\tilde{\vec{p}}_0^*$$

assume dipole excited by $\tilde{\vec{E}}_p$ (for example scattering of light in atmosphere)

$$\Rightarrow \tilde{\vec{p}}_0 = \frac{\tilde{\vec{P}}_0}{N} = \frac{\varepsilon_0(\varepsilon_r - 1)\tilde{E}_0\hat{e}_i}{N} \underset{[\varepsilon_r - 1 \approx 2(n-1)]}{\approx} \frac{2\varepsilon_0(n-1)\tilde{E}_0\hat{e}_i}{N}$$

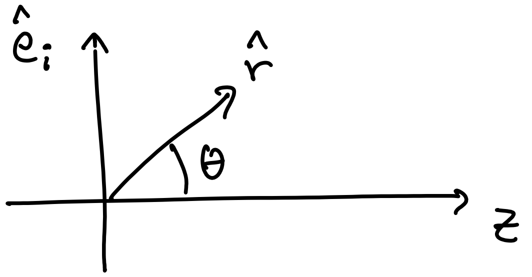
N ← nr of molecules per unit volume

$$\therefore \frac{d\sigma}{d\Omega} = \frac{k^4}{4\pi^2 N^2}(n-1)^2\left[1 - (\hat{e}_i\cdot\hat{r})^2\right]$$

Rayleigh scattering

$\hat{e}_i$ and $\hat{r}$ in same plane: $\hat{e}_i \cdot \hat{r} = \cos\left(\frac{\pi}{2} - \theta\right)$

$= \sin\theta$



$\hat{e}_i \perp \hat{r} \Rightarrow \hat{e}_i \cdot \hat{r} = 0$

average over polarisations

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{4\pi^2 N^2} (n-1)^2 \frac{1}{2} \left[1 + 1 - \sin^2\theta \right]$$

and

$$\sigma = \frac{2k^4}{3\pi N^2} (n-1)^2$$

$$\int \frac{1+x^2}{2} \, dx \, d\varphi = 2\pi \left[\frac{x}{2} + \frac{x^3}{6} \right]_{-1}^{1} = 2\pi \left[1 + \frac{1}{3} \right] = \frac{8\pi}{3}$$